FONDO PIZZOFALCONE

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INSTITUTIONES
CALCULI
DIFFERENTIALIS
VOL. II.

BIBLIOTECA NAZIONALE "VITTORIO EMANUELE III" NAPOLI

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CAPUT VI.

DE SUMMATIONE PROGRESSIONUM PER SERIES INFINITAS.

140.

Expreſſio generalis, quam in Capite praecedente pro termino ſummatorio cuiuſque ſeriei, cuius terminus generalis ſeu indici x reſpondens eſt $= z$, invenimus:

$$Sz = \int z dx + \frac{1}{2} z + \frac{\mathfrak{A} dz}{1.2 dx} - \frac{\mathfrak{B} d^3 z}{1.2.3.4 dx^3} + \frac{\mathfrak{C} d^5 z}{1.2 \ldots 6 dx^5} - \&c.$$

proprie inſervit ſeriebus ſummandis, quarum termini generales ſunt functiones quaecunque rationales integrae indicis x, quoniam his caſibus ad differentialia tandem evaneſcentia pervenitur. Sin autem z non fuerit eiuſmodi functio ipſius x, tum eius differentialia in infinitum progrediuntur, ſicque reſultat ſeries infinita ſummam ſeriei propoſitae exprimens, & quidem ad datum uſque terminum, cuius index eſt $= x$. Quocirca progreſſionis propoſitae in infinitum continuatae ſumma prodibit, ſi ponatur $x = \infty$; hocque pacto alia invenitur ſeries infinita priori aequalis.

141. Sin autem ponatur $x = 0$, tum expreſſio ſummam exhibens debet evaneſcere, uti iam annotavimus; quod niſi fiat, eiuſmodi quantitas conſtans ad ſummam addi vel inde auferri debet, ut huic conditioni ſatisfiat. Quo facto ſi ponatur $x = 1$, ſumma inventa praebebit terminum primum ſeriei: ſi $x = 2$, aggregatum primi & ſecundi; ſi $x = 3$, orietur aggregatum trium terminorum initialium ſeriei, & ita porro. His igitur caſibus, quia ſumma unius, vel duorum, vel trium, &c. terminorum eſt cognita, ſeriei infinitae, qua iſta ſumma exprimitur; valor innoteſcet; ex hocque fonte innumerabiles ſeries ſummari poterunt.

142. Quoniam, ſi eiuſmodi conſtans ſummae fuerit adie-

ſta, ut ea evaneſcat poſito $x=0$, tum ſumma omnibus reliquis caſibus, quicunque numeri pro x ſubſtituantur, ſatisfacit; manifeſtum eſt, dummodo ſummae inventae eiuſmodi quantitas conſtans adiiciatur, ut uno quodam caſu vera ſumma indicetur, tum omnibus reliquis caſibus veram ſummam prodire debere. Quare ſi ponendo $x=0$, non pateat, cuiuſmodi valorem expreſſio ſummae recipiat, neque igitur conſtans adiicienda hinc inveniri queat; tum alius quicunque numerus pro x ſtatui poterit, adiiciendaque conſtante effici ut debita ſumma indicetur: quod quomodo fieri debeat, ex ſequentibus magis fiet perſpicuum.

Conſideremus primum hanc progreſſionem harmonicam:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ \quad \cdots \quad +\frac{1}{x}=s,$$

cuius terminus generalis cum ſit $=\frac{1}{x}$, fiet $z=\frac{1}{x}$, & terminus ſummatorius s ita invenietur. Primo erit

$\int z dx=\int\frac{dx}{x}=lx$; deinde differentialia ita ſe habebunt:

$$\frac{dz}{dx}=-\frac{1}{x^2};\quad \frac{ddz}{2dx^2}=\frac{1}{x^3};\quad \frac{d^3z}{6dx^3}=-\frac{1}{x^4};$$

$\frac{d^4z}{24dx^4}=\frac{1}{x^5}$; $\frac{d^5z}{120dx^5}=-\frac{1}{x^6}$ &c. Hinc itaque erit:

$$s=lx+\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{\&c.}+\text{Conſt.}$$

Conſtans igitur hic addenda ex caſu $x=0$ non poteſt definiri. Ponatur ergo $x=1$, quia tum fit $s=1$, erit

$1=\frac{1}{2}-\frac{\mathfrak{A}}{2}+\frac{\mathfrak{B}}{4}-\frac{\mathfrak{C}}{6}+\frac{\mathfrak{D}}{8}+\text{Conſt.}$ unde fit iſta conſtans $=\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+$ &c. eritque ideo terminus ſummatorius quaeſitus:

$s=$

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$$s = lx + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \&c.$$
$$+ \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \&c.$$

143. Quoniam numeri Bernoulliani $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. conſtituunt ſeriem divergentem, hic valor conſtantis cognoſci nequit. Sin autem loco x ſubſtituatur numerus maior, atque ſumma totidem terminorum actu quaeratur, valor conſtantis commode inveſtigabitur. Ponatur in hunc finem $x = 10$, decemque primis terminis colligendis reperietur eorum ſumma =

$$2,9289682539682539 68$$

cui aequalis eſſe debet expreſſio ſummae, ſi in ea ponatur $x = 10$, quae fit:

$$l10 + \frac{1}{20} - \frac{\mathfrak{A}}{200} + \frac{\mathfrak{B}}{40000} - \frac{\mathfrak{C}}{6000000} + \frac{\mathfrak{D}}{800000000} - \&c. + C.$$

ſumto ergo pro $l10$ logarithmo hyperbolico denarii & loco $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. ſubſtitutis valoribus ſupra inventis, reperietur conſtans illa: $C = 0,5772156649015325$
qui numerus ergo exprimit ſummam ſeriei:

$$\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \frac{\mathfrak{E}}{10} - \&c.$$

144. Si pro x numeri non nimis magni ſubſtituantur, quia ſumma ſeriei facile actu invenitur, obtinebitur, ſumma ſeriei huius:

$$\frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \&c. = s - lx - C.$$

Sin autem x ſignificet numerum valde magnum, quia tum valor huius expreſſionis in infinitum excurrentis facile in fractionibus decimalibus aſſignatur, viciſſim ſumma ſeriei definietur. Ac primo quidem conſtat, ſi ſeries in infinitum continuetur, eius ſummam futuram eſſe infinite magnam; facto enim $x = \infty$ fit lx quoque infinitus; etſi ∞ ad x rationem infinite parvam teneat. Quo autem commodius ſumma

quotcunque terminorum ſeriei aſſignari queat, valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. in fractionibus decimalibus exprimamus:

$$\mathfrak{A} = 0,1666666666666$$
$$\mathfrak{B} = 0,0333333333333$$
$$\mathfrak{C} = 0,0238095238095$$
$$\mathfrak{D} = 0,0333333333333$$
$$\mathfrak{E} = 0,0757575757575$$
$$\mathfrak{F} = 0,2531135531135$$
$$\mathfrak{G} = 1,1666666666666$$
$$\mathfrak{H} = 7,0921568627451 \quad \&c.$$

unde ergo erit

$$\frac{\mathfrak{A}}{2} = 0,0833333333333$$
$$\frac{\mathfrak{B}}{4} = 0,0083333333333$$
$$\frac{\mathfrak{C}}{6} = 0,0039682539682$$
$$\frac{\mathfrak{D}}{8} = 0,0041666666666$$
$$\frac{\mathfrak{E}}{10} = 0,0075757575757$$
$$\frac{\mathfrak{F}}{12} = 0,0210927960928$$
$$\frac{\mathfrak{G}}{14} = 0,0833333333333$$
$$\frac{\mathfrak{H}}{16} = 0,4432598039216 \quad \&c.$$

EXEMPLUM I.

Invenire ſummam mille terminorum ſeriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

Po-

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Ponatur ergo $x = 1000$, & cum ſit $l10 = 2,3025850929940456840$ erit

$$\begin{array}{rrl}
 & lx = & 6,9077552789821 \\
 & \text{Conſt.} = & 0,5772156649015 \\
 & \frac{1}{2x} = & \underline{0,0005000000000} \\
 & & 7,4854709438836 \\
\text{ſubt.} & \frac{\mathfrak{A}}{2xx} = & \underline{0,0000000833333} \\
 & & 7,4854708605503 \\
\text{add.} & \frac{\mathfrak{B}}{4x^4} = & \underline{0,0000000000000} \\
\text{Ergo} & & 7,4854708605503
\end{array}$$

eſt ſumma quaeſita mille terminorum, qui nequidem ſeptem unitates cum ſemiſſe conficiunt.

EXEMPLUM II.

Invenire ſummam millies mille terminorum ſeriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$$

Quia eſt $x = 1000000$, erit $lx = 6.l10$, ergo

$$\begin{array}{rl}
lx = & 13,8155105579642 \\
\text{Conſt.} = & 0,5772156649015 \\
\frac{1}{2x} = & \underline{0,0000005000000} \\
 & 14,3927267228657 = \text{ſummae quaeſitae.}
\end{array}$$

145. Si ergo pro x ſtatuatur numerus vehementer magnus, ſumma ſatis exacte invenitur ex ſolo primo termino lx conſtante C aucto: unde egregia corollaria deduci poſſunt. Sic ſi x fuerit numerus vehementer magnus, ponaturque:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{x} = s$$

$$\& \; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots + \frac{1}{x} + \ldots \frac{1}{x+y} = t$$

quia

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quia eſt proxime $s = lx + C$, & $t = l(x+y) + C$; erit $t - s = l(x+y) - lx = l\frac{x+y}{x}$, ideoque hic logarithmus proxime per ſeriem harmonicam finito terminorum numero conſtantem exprimetur hoc modo:

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \quad \dots \quad + \frac{1}{x+y}.$$

Accuratius autem hic logarithmus exhibebitur, ſi ſuperiores ſummae s & t exactius capiantur. Sic cum ſit

$$s = lx + C + \frac{1}{2x} - \frac{1}{12xx}, \qquad \&$$

$$t = l(x+y) + C + \frac{1}{2(x+y)} - \frac{1}{12(x+y)^2}; \quad \text{erit}$$

$$t - s = l\frac{x+y}{x} - \frac{1}{2x} + \frac{1}{2(x+y)} + \frac{1}{12xx} - \frac{1}{12(x+y)^2}, \quad \text{ideoque}$$

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+y} + \frac{1}{2x} - \frac{1}{2(x+y)} - \frac{1}{12xx} + \frac{1}{12(x+y)^2}$$

Sin autem ſit numerus tam magnus, ut bini termini ultimi reiici queant, erit proxime:

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+y} + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+y}\right)$$

Ex hac quoque ſerie harmonica derivare poterimus ſummam huius ſeriei, in qua tantum numeri impares occurrunt:

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \quad \dots \quad + \frac{1}{2x+1}.$$

Cum enim omnibus terminis capiendis ſit:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \quad \dots\dots \quad + \frac{1}{2x} + \frac{1}{2x+1} =$$

$$l(2x+1) + C + \frac{1}{2(2x+1)} - \frac{\mathfrak{A}}{2(2x+1)^2} + \frac{\mathfrak{B}}{4(2x+1)^4} - \frac{\mathfrak{C}}{6(2x+1)^6} + \&c.$$

termi-

terminorum vero ordine parium:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\quad\ldots\ldots\ldots\quad+\frac{1}{2x}$$

ſumma ſit ſemiſſis ſuperioris nempe:

$$\tfrac{1}{2}C+\tfrac{1}{2}lx+\frac{1}{4x}-\frac{\mathfrak{A}}{4x^2}+\frac{\mathfrak{B}}{8x^4}-\frac{\mathfrak{C}}{12x^6}+\frac{\mathfrak{D}}{16x^8}-\&c.$$

erit hac ſerie ab illa ablata:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\quad\ldots\ldots\ldots\quad+\frac{1}{2x+1}=$$

$$\tfrac{1}{2}C+l\frac{2x+1}{\sqrt{x}}+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\&c.$$

$$-\frac{1}{4x}+\frac{\mathfrak{A}}{4x^2}-\frac{\mathfrak{B}}{8x^4}+\&c.$$

146. Poteſt vero etiam per eandem expreſſionem generalem ſumma cuiuſque ſeriei harmonicae inveniri; ſit enim;

$$\frac{1}{m+n}+\frac{1}{2m+n}+\frac{1}{3m+n}+\frac{1}{4m+n}+\quad\ldots\quad+\frac{1}{mx+n}=s,$$

quia eſt terminus generalis $z=\frac{1}{mx+n}$, erit:

$$\int zdx=\frac{1}{m}l(mx+n);\quad \frac{dz}{dx}=-\frac{m}{(mx+n)^2}$$

$$\frac{ddz}{2dx^2}=\frac{mm}{(mx+n)^3};\quad \frac{d^3z}{6dx^3}=-\frac{m^3}{(mx+n)^4}$$

$$\frac{d^4z}{24dx^4}=\frac{m^4}{(mx+n)^5};\quad \frac{d^5z}{120dx^5}=-\frac{m^5}{(mx+n)^6}\ \&c.$$

Ex his ergo reperitur:

$$s=D+\frac{1}{m}l(mx+n)+\frac{1}{2(mx+n)}-\frac{\mathfrak{A}m}{2(mx+n)^2}+\frac{\mathfrak{B}m^3}{4(mx+n)^4}$$

$$-\frac{\mathfrak{C}m^5}{6(mx+n)^6}+\frac{\mathfrak{D}m^7}{8(mx+n)^8}-\&c.$$

Po-

Poſito ergo $x = 0$, fiet conſtans illa addenda:

$$D = -\frac{1}{m}\,ln - \frac{1}{2n} + \frac{\mathfrak{A}m}{2n^2} - \frac{\mathfrak{B}m^3}{4n^4} + \frac{\mathfrak{C}m^5}{6n^6} - \&c.$$

147. Si vero ſit $n = 0$, quoniam ſeriei:

$$\frac{1}{m} + \frac{1}{2m} + \frac{1}{3m} + \frac{1}{4m} + \ldots\ldots\ldots + \frac{1}{mx}$$

ſumma eſt $= \frac{1}{m}C + \frac{1}{m}lx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2mx^2} + \frac{\mathfrak{B}}{4mx^4} - \&c.$

at vero huius ſeriei:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\ldots + \frac{1}{mx}$$

ſumma eſt $= C + lmx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \&c.$

ſi ab hac ſerie illa m vicibus ſumta ſubtrahatur ut prodeat haec ſeries:

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{m} + \ldots + \frac{1}{2m} + \ldots + \frac{1}{3m} + \ldots + \frac{1}{mx}$$
$$\quad - \frac{m}{m} \qquad - \frac{m}{2m} \qquad - \frac{m}{3m} \qquad - \frac{m}{mx}$$

eius ſumma erit $= lm + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \&c.$

$$- \frac{1}{2x} + \frac{\mathfrak{A}}{2xx} - \frac{\mathfrak{B}}{4x^4} + \&c.$$

atque ſi ſtatuatur $x = \infty$ ſumma erit $= lm$. Hinc pro m ponendo numeros 2, 3, 4, &c. erit:

$l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \&c.$
$l3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9}\ \&c.$
$l4 = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \&c.$
$l5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{4}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{4}{10}\ \&c.$
&c.

148. Relicta autem ſerie harmonica progrediamur ad ſeriem quadratorum reciprocam, ſitque:

$s =$

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\dots + \frac{1}{xx},$$

in qua cum ſit terminus generalis $z = \frac{1}{xx}$, erit

$\int z dx = -\frac{1}{x}$, & differentialia ipſius z ita ſe habebunt

$$\frac{dz}{2dx} = -\frac{1}{x^3};\ \frac{ddz}{2.3dx^2} = \frac{1}{x^4};\ \frac{d^3z}{2.3.4dx^3} = -\frac{1}{x^5};\ \text{\&c.}$$

unde erit ſumma

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \text{\&c.}$$

in qua conſtans addenda C ex uno caſu, quo ſumma conſtat, eſt definienda. Ponamus ergo $x = 1$, quia fit $s = 1$ debet eſſe: $C = 1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{\&c.}$ quae ſeries autem cum ſit maxime divergens, valorem conſtantis C non oſtendit. Quia autem ſupra demonſtravimus ſummam huius ſeriei in infinitum continuatae eſſe $= \frac{\pi\pi}{6}$: facto $x = \infty$, ſi ponatur $s = \frac{\pi\pi}{6}$, fiet $C = \frac{\pi\pi}{6}$; ob reliquos terminos omnes evaneſcentes. Erit ergo

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{\&c.} = \frac{\pi\pi}{6}.$$

149. Sin autem ſumma huius ſeriei cognita non fuiſſet, valor conſtantis illius C ex alio quopiam caſu, quo ſumma actu eſt inventa, determinari deberet. Hunc in finem ponamus $x = 10$, atque decem terminis actu addendis reperietur:

$s = 1,5497677311665406\,90$ tum eſt

add. $\frac{1}{x} = 0,1$

ſubtr. $\frac{1}{2xx} = 0,005$

$1,6447677311665406\,90$

$$\begin{array}{llcl}
 & & & 1,6447677311665406\,90 \\
\text{add.} & \dfrac{\mathfrak{A}}{x^3} & = & 0,000166666666666666 \\
\hline
 & & & 1,644934397833207356 \\
\text{ſubtr.} & \dfrac{\mathfrak{B}}{x^5} & = & 0,000000333333333333 \\
\hline
 & & & 1,644934064499874023 \\
\text{add.} & \dfrac{\mathfrak{C}}{x^7} & = & 0,000000002380952381 \\
\hline
 & & & 1,644934066880826404 \\
\text{ſubtr.} & \dfrac{\mathfrak{D}}{x^9} & = & 0,000000000033333333 \\
\hline
 & & & 1,644934066847493071 \\
\text{add.} & \dfrac{\mathfrak{E}}{x^{11}} & = & 0,000000000000757575 \\
\hline
 & & & 1,644934066848250646 \\
\text{ſubtr.} & \dfrac{\mathfrak{F}}{x^{13}} & = & 0,000000000000025311 \\
\hline
 & & & 1,644934066848225335 \\
\text{add.} & \dfrac{\mathfrak{G}}{x^{15}} & = & 0,000000000000001166 \\
 & & & \\
\text{ſubtr.} & \dfrac{\mathfrak{H}}{x^{17}} & = & 71 \\
\hline
 & & & 1,644934066848226430 = C.
\end{array}$$

Hicque numerus ſimul eſt valor expreſſionis $\frac{\pi\pi}{6}$, quemadmodum ex valore ipſius π cognito calculum inſtituenti patebit. Unde ſimul intelligitur, etiamſi ſeries $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. divergat, tamen hoc modo veram prodire ſummam.

150. Sit nunc $z = \frac{1}{x^3}$; atque

$$s = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots\ldots + \frac{1}{x^3},$$ quia eſt

$$\int z\,dx = -\frac{1}{2xx};\quad \frac{dz}{1.2.3\,dx} = -\frac{1}{2x^4};\quad \frac{ddz}{1.2.3.4\,dx^2} = \frac{1}{2x^5}$$

$$\frac{d^3z}{1.2\ldots5\,dx^3} = -\frac{1}{2x^6};\quad \frac{d^5z}{1.2\ldots7\,dx^5} = -\frac{1}{2x^8};$$ &c. erit

$s =$

$$s = C - \frac{1}{2xx} + \frac{1}{2x^3} - \frac{3\mathfrak{A}}{2x^4} + \frac{5\mathfrak{B}}{2x^6} - \frac{7\mathfrak{C}}{2x^8} + \&c.$$

hincque poſito $x = 1$, ob $s = 1$, fiet:

$$C = 1 + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{3}{2}\mathfrak{A} - \tfrac{5}{2}\mathfrak{B} + \tfrac{7}{2}\mathfrak{C} - \tfrac{9}{2}\mathfrak{D} + \&c.$$

atque iſte valor ipſius C ſimul oſtendet ſummam ſeriei propoſitae in infinitum continuatae. Quoniam vero ſummae poteſtatum imparium non aeque ac parium conſtant, iſte ipſius C valor ex cognita ſumma aliquot terminorum definiri debet. Sit ergo $x = 10$, erit:

$$C = s + \frac{1}{2xx} - \frac{1}{2x^3} + \frac{3\mathfrak{A}}{2x^4} - \frac{5\mathfrak{B}}{2x^6} + \frac{7\mathfrak{C}}{2x^8} - \&c.$$

Eſt vero ad computum facilius inſtituendum:

$$\frac{3\mathfrak{A}}{2} = 0,250000000000$$

$$\frac{5\mathfrak{B}}{2} = 0,083333333333$$

$$\frac{7\mathfrak{C}}{2} = 0,083333333333$$

$$\frac{9\mathfrak{D}}{2} = 0,150000000000$$

$$\frac{11\mathfrak{E}}{2} = 0,416666666666$$

$$\frac{13\mathfrak{F}}{2} = 1,645238095238$$

$$\frac{15\mathfrak{G}}{2} = 8,750000000000$$

$$\frac{17\mathfrak{H}}{2} = 60,283333333333 \quad \&c.$$

 Hinc

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Hinc ergo fient termini ad s addendi:

$$\frac{1}{2xx} = 0,005000000000000000$$

$$\frac{3\mathfrak{A}}{2x^4} = 0,000025000000000000$$

$$\frac{7\mathfrak{C}}{2x^8} = 0,000000000833333333$$

$$\frac{11\mathfrak{E}}{2x^{12}} = 0,000000000000416666$$

$$\frac{15\mathfrak{G}}{2x^{16}} = \underline{0,000000000000000875}$$

$$0,005025000833750875$$

termini autem ſubtrahendi ſunt:

$$\frac{1}{2x^3} = 0,000500000000000000$$

$$\frac{5\mathfrak{B}}{2x^6} = 0,000000083333333333$$

$$\frac{9\mathfrak{D}}{2x^{10}} = 0,000000000001500000$$

$$\frac{13\mathfrak{F}}{2x^{14}} = 0,000000000000016452$$

$$\frac{17\mathfrak{H}}{2x^{18}} = \underline{0,000000000000000060}$$

$$0,000500083348349845$$

$$\text{ab}: \underline{0,005025000833750875}$$

$$0,004524917485401030$$

$$s = \underline{1,197531985674193251}$$

$$C = 1,262056903159594281.$$

151. Si hoc modo ulterius progrediamur, inveniemus ſummas omnium ſerierum poteſtatum reciprocarum in fractionibus decimalibus expreſſas:

1 +

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.=1{,}6449340668482264=\frac{2\,\mathfrak{A}}{1.\,2}\,\pi^2$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.=1{,}2020569031595942$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.=1{,}0823232337111381=\frac{2^3\,\mathfrak{B}}{1.\,2.\,3.\,4}\,\pi^4$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c.=1{,}0369277551068632$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c.=1{,}0173430619844491=\frac{2^5\,\mathfrak{C}}{1.\,2\ldots\,6}\,\pi^6$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\&c.=1{,}0083492773866018$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c.=1{,}0040773561979443=\frac{2^7\,\mathfrak{D}}{1.\,2\ldots\,8}\,\pi^8$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\&c.=1{,}0020083928260822$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\&c.=1{,}0009945751278180=\frac{2^9\,\mathfrak{E}}{1.2\ldots 10}\,\pi^{10}$$

$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\&c.=1{,}0004941886041094$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\&c.=1{,}0002460865533080=\frac{2^{11}\,\mathfrak{F}}{1.2\ldots 12}\,\pi^{12}$$

$$1+\frac{1}{2^{13}}+\frac{1}{3^{13}}+\frac{1}{4^{13}}+\&c.=1{,}0001227233475857$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\&c.=1{,}0000612481350587=\frac{2^{13}\,\mathfrak{G}}{1.2\ldots 14}\,\pi^{14}$$

$$1+\frac{1}{2^{15}}+\frac{1}{3^{15}}+\frac{1}{4^{15}}+\&c.=1{,}0000305882363070$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\&c.=1{,}0000152822594086=\frac{2^{15}\,\mathfrak{H}}{1.2\ldots 16}\,\pi^{16}$$

&c.

152. Ex his ergo viciſſim ſummae illarum ſerierum infinitarum numeris Bernoullianis conſtantium exhiberi poterunt. Erit enim:

$$1+0-\tfrac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\&c.=0,57721\ \&c.$$

$$1+1-\tfrac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\&c.=\frac{2\mathfrak{A}}{1.2}\pi^2$$

$$1+\tfrac{1}{2}-\tfrac{1}{2}+\frac{3\mathfrak{A}}{2}-\frac{5\mathfrak{B}}{2}+\frac{7\mathfrak{C}}{2}-\frac{9\mathfrak{D}}{2}+\&c.=1,2020\ \&c.$$

$$1+\tfrac{1}{3}-\tfrac{1}{2}+\frac{3.4\mathfrak{A}}{2.3}-\frac{5.6\mathfrak{B}}{2.3}+\frac{7.8\mathfrak{C}}{2.3}-\frac{9.10\mathfrak{D}}{2.3}+\&c.=\frac{2^3\mathfrak{B}}{1.2.3.4}\pi^4$$

$$1+\tfrac{1}{4}-\tfrac{1}{2}+\frac{3.4.5\mathfrak{A}}{2.3.4}-\frac{5.6.7.\mathfrak{B}}{2.3.4}+\frac{7.8.9\mathfrak{C}}{2.3.4}-\frac{9.10.11\mathfrak{D}}{2.3.4.}+\&c.=1,0369\ \&c.$$

$$1+\tfrac{1}{5}-\tfrac{1}{2}+\frac{3.4.5.6\mathfrak{A}}{2.3.4.5}-\frac{5.6.7.8\mathfrak{B}}{2.3.4.5}+\frac{7.8.9.10\mathfrak{C}}{2.3.4.5.}-\&c.=\frac{2^5\mathfrak{C}}{1.2\ldots6}\pi^6$$

Harum ergo ſerierum alternae ope quadraturae circuli ſummari poſſunt; a quanam vero quantitate tranſcendente reliquae pendeant, adhuc non conſtat: neque enim ad poteſtates ipſius π exponentes impares habentes revocari poſſunt, ita ut coefficientes eſſent numeri rationales. Quo autem ſaltem proxime appareat, quales futuri ſint coefficientes poteſtatum ipſius π pro exponentibus imparibus, tabellam ſequentem adiunximus:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.\ \text{in infin.}=\frac{\pi}{0,0000}=\infty$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.\quad..\quad=\frac{\pi^2}{6,0000}\quad\text{vere}$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.\quad..\quad=\frac{\pi^3}{25,79436}\quad\text{prox.}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.\quad..\quad=\frac{\pi^4}{90,00000}\quad\text{vere}$$

1 +

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c. \;.\;.\; = \frac{\pi^5}{295,1215} \quad \text{prox.}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c. \;.\;.\; = \frac{\pi^6}{945,000} \quad \text{vere}$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\&c. \;.\;.\; = \frac{\pi^7}{2995,286} \quad \text{prox.}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c. \;.\;.\; = \frac{\pi^8}{9450,000} \quad \text{vere}$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\&c. \;.\;.\; = \frac{\pi^9}{29749,35} \quad \text{prox.}$$

&c.

153. Ex hoc fonte ſeries numerorum Bernoullianorum

1 2 3 4 5 6 7 8 9
𝔄, 𝔅, ℭ, 𝔇; 𝔈 𝔉, 𝔊, ℌ, ℑ, &c.

quantumvis irregularis videatur interpolari, ſeu termini in medio binorum quorumcunque conſtituti aſſignari poterunt: ſi enim terminus medium interiacens inter primum 𝔄 & ſecundum 𝔅, ſeu indici $1\frac{1}{2}$ reſpondens fuerit $=p$; erit utique:

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\&c. \quad = \quad \frac{2^2 p}{1.2.3}\pi^3$$

ideoque $\quad p=\frac{3}{2\pi^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\&c.\right)=0,05815227$

Simili modo ſi terminus inter 𝔅 & ℭ medium interiacens ſeu indicem habens $2\frac{1}{2}$ ponatur $=q$, quia erit:

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\&c. \quad = \quad \frac{2^4 q}{1.2.3.4.5}\pi^5$$

fiet $\quad q=\frac{15}{2\pi^5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\&c.\right)=0,02541327$

Si ergo iſtarum ſerierum, in quibus exponentes poteſtatum ſunt numeri impares, ſummae exhiberi poſſent, tum quoque ſeries numerorum Bernoullianorum interpolari poſſet.

154. Ponamus nunc $z=\frac{1}{mm+xx}$, & quaeratur

ſumma huius ſeriei:

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+9} + \ \cdot\ \cdot\ \cdot\ \cdot\ + \frac{1}{nn+xx}.$$

Quia eſt $\int z dx = \int \frac{dx}{nn+xx}$; erit $\int z dx = \frac{1}{n}\, A \tang \frac{x}{n}$.

Ponatur $A \cot \frac{x}{n} = u$, erit $\int z dx = \frac{1}{n}\left(\frac{\pi}{2} - u\right)$, &

$\frac{x}{n} = \cot u = \frac{\cos u}{\sin u}$, & $\frac{nn+xx}{nn} = \frac{1}{\sin u^2}$, & $z = \frac{\sin u^2}{nn}$,

& $\frac{dx}{n} = -\frac{du}{\sin u^2}$, unde fit $du = -\frac{dx \sin u^2}{n}$.

Hinc differentialia ipſius z invenientur hoc modo:

$$dz = \frac{2\, du \sin u . \cos u}{nn} = -\frac{dx \sin u^2 . \sin 2u}{n^3} \ \& \ \frac{dz}{dx} = -\frac{\sin u^2 . \sin 2u}{n^3}$$

$$\frac{ddz}{2dx} = -\frac{du(\sin u \cos u \sin 2u + \sin u^2 . \cos 2u)}{n^3} = \frac{dx \sin u^3 . \sin 3u}{n^4}$$

$$\& \quad \frac{ddz}{2dx^2} = \frac{\sin u^3 . \sin 3u}{n^4}.$$

Simili modo erit, uti iam ſupra pro eodem caſu invenimus:

$$\frac{d^3 z}{2.3dx^3} = -\frac{\sin u^4 . \sin 4u}{n^5}; \quad \frac{d^4 z}{2.3.4dx^4} = \frac{\sin u^5 . \sin 5u}{n^6} \ \&c.$$

ex quibus formabitur ſumma quaeſita:

$$s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\sin u . \sin u}{2nn} - \frac{\mathfrak{A}}{2} . \frac{\sin u^2 . \sin 2u}{n^3} + \frac{\mathfrak{B}}{4} . \frac{\sin u^4 . \sin 4u}{n^5}$$
$$- \frac{\mathfrak{C}}{6} . \frac{\sin u^6 . \sin 6u}{n^7} + \frac{\mathfrak{D}}{8} . \frac{\sin u^8 . \sin 8u}{n^9} - \&c. \quad + \text{Conſt.}$$

Si hic ad conſtantem determinandam ponatur $x = 0$, quo fiat $s = 0$, erit $\cot u = 0$, ideoque u angulus $90°$, ac propterea $\sin u = 1$, $\sin 2u = 0$, $\sin 4u = 0$, $\sin 6u = 0$, &c. videtur ergo fore $0 = \frac{\pi}{2n} - \frac{\pi}{2n} + \frac{1}{2nn} + C$, hinc $C = -\frac{1}{2nn}$:

at

at vero notandum eſt, etiamſi reliqui termini evaneſcant, tamen quia coefficientes 𝔄, 𝔅, ℭ, &c. tandem in infinitum excreſcunt, eorum ſummam poſſe eſſe finitam.

155. Ad hanc ergo conſtantem rite determinandam ponamus eſſe $x = \infty$, ſummam enim huius ſeriei in infinitum excurrentis ſupra iam in introductione definivimus, oſtendimuſque eſſe eam $= -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi}-1)}$. Poſito autem $x = \infty$, fiet $u = 0$, ideoque ſin $u = 0$, ſimulque ſinus omnium arcuum multiplorum evaneſcent. Cum autem in hac ſerie poteſtates ipſius ſin u creſcant, divergentia ſeriei impedire nequit, quominus valor ſeriei hoc caſu evaneſcat. Fiet ergo $s = \frac{\pi}{2n} + C$; unde erit

$$\frac{\pi}{2n} + C = -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi}-1)}, \quad \&$$

$C = -\frac{1}{2nn} + \frac{\pi}{n(e^{2n\pi}-1)}$. Quare ſumma ſeriei quaeſita erit

$$s = \frac{\pi}{2n} - \frac{u}{n} - \frac{\operatorname{cof} u^2}{2nn} - \frac{\mathfrak{A}}{2}\cdot\frac{\operatorname{fin} u^2 \cdot \operatorname{fin} 2u}{n^3} + \frac{\mathfrak{B}}{4}\cdot\frac{\operatorname{fin} u^4 \cdot \operatorname{fin} 4u}{n^5} - \frac{\mathfrak{C}}{6}\cdot\frac{\operatorname{fin} u^6 \cdot \operatorname{fin} 6u}{n^7} + \&c. + \frac{\pi}{n(e^{2n\pi}-1)}.$$

Ubi notandum eſt, ſi n fuerit numerus mediocriter magnus, ultimum terminum $\frac{\pi}{n(e^{2n\pi}-1)}$ tantopere fieri exiguum, ut negligi queat.

156. Ponamus eſſe $x = n$, ita ut denotet:

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+9} + \ldots\ldots + \frac{1}{nn+nn}.$$

Tum vero erit $\cot u = 1$, & $u = 45° = \frac{\pi}{4}$. Quamobrem

 habe-

habebitur $\text{ſin}\, u = \frac{1}{\sqrt{2}}$; $\text{ſin}\, 2u = 1$; $\text{ſin}\, 4u = 0$; $\text{ſin}\, 6u = -1$; $\text{ſin}\, 8u = 0$; $\text{ſin}\, 10u = 1$; &c. Hancobrem erit:

$$s = \frac{\pi}{4n} - \frac{1}{4nn} - \frac{\mathfrak{A}}{2.2n^3} + \frac{\mathfrak{C}}{6.8n^7} - \frac{\mathfrak{E}}{10.2^5 n^{11}} + \frac{\mathfrak{G}}{14.2^7 n^{15}} - \&c. + \frac{\pi}{n(e^{2n\pi}-1)},$$

in qua expreſſione tantum numeri alterni ex Bernoullianis occurrunt. Si igitur valor ipſius s per computum actu inſtitutum iam ſuerit inventus, hinc quantitas π definiri poterit, erit enim:

$$\pi = 4ns + \frac{1}{n} + \frac{\mathfrak{A}}{1.n^2} - \frac{\mathfrak{C}}{3.2^2 n^6} + \frac{\mathfrak{E}}{5.2^4 n^{10}} - \frac{\mathfrak{G}}{7.2^6.n^{14}} + \&c. - \frac{\pi}{e^{2n\pi}-1}.$$

Etſi enim in termino ultimo ineſt π, tamen quia is tantopere eſt parvus, ſufficit valorem ipſius π proxime noſſe.

EXEMPLUM. Sit $n = 5$; erit:

$$s = \frac{1}{26} + \frac{1}{29} + \frac{1}{34} + \frac{1}{41} + \frac{1}{50}$$

qui termini actu additi dabunt:

$$s = 0,1467463056905494 94$$

unde erunt termini illi;

$$4ns = 2,9349261138109888$$

$$\frac{1}{n} = 0,2$$

$$\frac{\mathfrak{A}}{nn} = \underline{0,0066666666666666}$$

$$3,14159278047765654$$

$$\frac{\mathfrak{C}}{3.2^2.n^6} = \underline{0,0000001269841269 8}$$

$$3,14159265349352956$$

$\mathfrak{E}$

$$3,14159265349352956$$

$$\frac{\mathfrak{E}}{5.2^4.n^{10}} = 0,00000000009696969$$

$$3,14159265359049925$$

$$\frac{\mathfrak{G}}{7.2^6.n^{14}} = 0,00000000000042666$$

$$3,14159265359007259$$

$$\frac{\mathfrak{I}}{9.2^8.n^{18}} = 625$$

$$3,14159265359007884$$

Hic valor iam tam prope ad veritatem accedit, ut mirandum ſit tam levi calculo eouſque perveniri poſſe. Eſt vero haec expreſſio aliquantillum iuſto maior, ſubtrahi enim adhuc inde debet $\frac{4\pi}{e^{2n\pi}-1}$, cuius valor, dummodo π prope ſit cognitum, exhiberi poteſt; quod per logarithmos ita expedietur.

Quia eſt $\pi le = 1,3643763538$

erit $le^{2n\pi} = 10\pi le = 13,6437635$.

Cum iam ſit $\frac{4\pi}{e^{2n\pi}-1} = \frac{4\pi}{e^{2n\pi}} + \frac{4\pi}{e^{4n\pi}} + \&c.$

facile intelligitur ad noſtrum calculum ſufficere primum terminorum ſumſiſſe. Augeamus ergo characteriſticam numero 17, quia habemus totidem figuras decimales, erit:

$$l\pi = 17,4971498$$

$$l4 = 0,6020600$$

$$18,0992098$$

Subtr. $le^{2n\pi} = 13,6437635$

$$4,4554463$$

Ergo $\frac{4\pi}{e^{2n\pi}} = 28539$ ſubtrahatur

ab $3,14159265359007884$ erit

$$\pi = 3,14159265358979345$$

quae expreſſio in figura demum penultima a veritate recedit;

 quod

quod mirandum non eſt, cum adhuc terminum $\frac{\mathfrak{L}}{11.2^{10}.\pi^{22}}$, qui dat 22, ſubtrahere debuiſſemus, ſicque ne ultima quidem figura aberraſſet. Ceterum intelligitur, ſi pro n maiorem numerum uti 10 aſſumſiſſemus, tum facili negotio peripheriam π ad 25 pluresque figuras inveniri potuiſſe.

157. Ponamus nunc quoque pro z functiones tranſcendentes ipſius x, ſitque $z = lx$, ſumendo logarithmos hyperbolicos, quoniam vulgares facile eo revocantur; ſitque:

$$s = l1 + l2 + l3 + l4 + \ldots + lx.$$

Quia igitur eſt $z = lx$, erit $\int z dx = xlx - x$, huius enim differentiale dat $dxlx$. Deinde eſt

$$\frac{dz}{dx} = \frac{1}{x};\ \frac{ddz}{dx^2} = -\frac{1}{x^2};\ \frac{d^3z}{1.2dx^3} = \frac{1}{x^3};\ \frac{d^5z}{1.2.3.4dx^5} = \frac{1}{x^5};\ \&c.$$

Hinc itaque concluditur fore:

$$s = xlx - x + \tfrac{1}{2}lx + \frac{\mathfrak{A}}{1.2x} - \frac{\mathfrak{B}}{3.4x^3} + \frac{\mathfrak{C}}{5.6x^5} - \frac{\mathfrak{D}}{7.8x^7} + \&c. + \text{Conſt.}$$

Haec autem conſtans ponendo $x = 1$, quia fit $s = l1 = 0$ ita definietur, ut ſit: $C = 1 - \frac{\mathfrak{A}}{1.2} + \frac{\mathfrak{B}}{3.4} - \frac{\mathfrak{C}}{5.6} + \frac{\mathfrak{D}}{7.8} - \&c.$ quae ſeries ob nimiam divergentiam eſt inepta ad valorem ipſius C ſaltem proxime eruendum.

158. Non ſolum autem proximum ſed etiam ipſum verum valorem ipſius C inveniemus, ſi conſideremus expreſſionem Wallisianam pro valore ipſius π inventam, atque in introductione demonſtratam: quae erat:

$$\frac{\pi}{2} = \frac{2.\,2.\,4.\,4.\,6.\,6.\,8.\,8.\,10.\,10.\,12.\ \&c.}{1.\,3.\,3.\,5.\,5.\,7.\,7.\,9.\,9.\,11.\,11.\ \&c.}$$

hinc enim logarithmis ſumendis erit:

$$l\pi - l2 = 2l2 + 2l4 + 2l6 + 2l8 + 2l10 + \&c.$$
$$- l1 - 2l3 - 2l5 - 2l7 - 2l9 - 2l11 - \&c.$$

Ponamus ergo in ſerie aſſumta $x = \infty$ & cum ſit:

$l1$

$$l1+l2+l3+l4+\dots+lx = C+(x+\tfrac{1}{2})lx-x$$

erit $l1+l2+l3+l4+\dots+l2x = C+(2x+\frac{1}{2})l2x-2x$

& $l2+l4+l6+l8+\dots+l2x = C+(x+\frac{1}{2})lx+xl2-x$

hinc $l1+l3+l5+l7+\dots+l(2x-1) = xlx+(x+\frac{1}{2})l2-x$

Cum igitur ſit:

$$l\frac{\pi}{2} = 2l2+2l4+2l6+\dots+2l2x-l2x$$
$$-2l1-2l3-2l5-\dots-2l(2x-1)$$

poſito $x=\infty$, erit:

$$l\frac{\pi}{2} = 2C+(2x+1)lx+2xl2-2x-l2-lx$$
$$-2xlx-(2x+1)l2+2x,$$

ideoque $l\frac{\pi}{2} = 2C-2l2$, ergo $2C = l2\pi$, & $C = \frac{1}{2}l2\pi$,

unde in fractionibus decimalibus reperitur:

$$C = 0,9189385332046727417803297$$

atque ſimul ſequens ſeries ſummatur:

$$1-\frac{\mathfrak{A}}{1.2}+\frac{\mathfrak{B}}{3.4}-\frac{\mathfrak{C}}{5.6}+\frac{\mathfrak{D}}{7.8}-\frac{\mathfrak{E}}{9.10}+\&c. = \tfrac{1}{2}l2\pi.$$

159. Cognita nunc iſta conſtante $C=\frac{1}{2}l2\pi$, ſumma quotcunque logarithmorum ex hac ſerie $l1+l2+l3+$ &c. exhiberi poteſt. Si enim ponatur:

$$s = l1+l2+l3+l4+\dots+lx, \text{ erit}$$

$$s = \tfrac{1}{2}l2\pi+(x+\tfrac{1}{2})lx-x+\frac{\mathfrak{A}}{1.2x}-\frac{\mathfrak{B}}{3.4x^3}+\frac{\mathfrak{C}}{5.6x^5}-\frac{\mathfrak{D}}{7.8x^7}+\&c.$$

ſi quidem logarithmi propoſiti fuerint hyperbolici; ſin autem proponantur logarithmi vulgares, tum in terminis $\frac{1}{2}l2\pi+(x+\frac{1}{2})lx$ pro $l2\pi$ & lx ſummi debebunt logarithmi vulgares, reliqui autem ſeriei termini $-x+\frac{\mathfrak{A}}{1.2x}-\frac{\mathfrak{B}}{3.4x^3}+\&c.$ multiplicari debent per $0,434294481903251827 = n$.

Erit igitur hoc caſu pro logarithmis vulgaribus:

$l\pi$

$$l\pi = 0,497149872694133854351268$$
$$l2 = 0,3010299956639811952137388$$
$$l2\pi = 0,7981798683581150495650006$$
$$\tfrac{1}{2}l2\pi = 0,3990899341790575247825003$$

EXEMPLUM

Quaeratur aggregatum mille logarithmorum tabularium

$$s = l1 + l2 + l3 + \;\ldots\ldots\; + l1000.$$

Erit ergo $x = 1000$, & $lx = 3,0000000000000$

uude fit $xlx = 3000,0000000000000$

$\tfrac{1}{2}lx = 1,5000000000000$

$\tfrac{1}{2}l2\pi = 0,3990899341790$

$3001,8990899341790$

ſubtr. $nx = 434,2944819032518$

$2567,6046080309272$

Deinde eſt $\dfrac{n\mathfrak{A}}{1.2x} = 0,0000361912068$

ſubtr. $\dfrac{n\mathfrak{B}}{3.4x^3} = 0,0000000000012$

$0,0000361912056$

addatur $2567,6046080309272$

ſumma quaeſita $s = 2567,6046442221328.$

Cum igitur s ſit logarithmus producti numerorum:

$$1.\; 2.\; 3.\; 4.\; 5.\; 6.\; \ldots\ldots\; 1000$$

patet hoc productum, ſi actu multiplicetur, conſtare ex 2568 figuris, atque notas a leva initiales fore 4023872 quas inſuper 2561 figurae ſequentur.

160. Ope ergo huius logarithmorum ſummationis producta ex quotcumque factoribus, qui ſecundum numeros naturales procedunt, proxime aſſignari poterunt. Huc potiſſimum referri poteſt problema, quo quaeritur uncia media ſeu maxima in poteſtate binomii quacumque $(a+b)^m$; ubi quidem notandum eſt, ſi m ſit numerus impar, binas dari medias inter ſe aequales, quae iunctim ſumtae praebeant unciam mediam

diam in poteſtate ſequente pari. Quare cum uncia maxima in quaque poteſtate pari ſit duplo maior quam uncia media in poteſtate praecedente impari, ſufficiet pro poteſtatibus paribus unciam mediam maximam determinaſſe. Sit igitur $m = 2n$, & uncia media ita exprimetur ut ſit:

$$\frac{2n(2n-1)(2n-2)(2n-3)\;.\;.\;.\;.\;(n+1)}{1.\;2.\;3.\;4.\;.\;.\;.\;.\;.\;.\;.\;.\;n}$$

Vocetur iſta uncia media quae quaeritur $= u$, atque eo hoc modo repraeſentari poterit, ut ſit:

$$u = \frac{1.\;2.\;3.\;4.\;5.\;.\;.\;.\;.\;.\;.\;2n}{(1.\;2.\;3.\;4.\;.\;.\;.\;n)^2}$$

ſumtiſque logarithmis erit:

$$lu = l1 + l2 + l3 + l4 + l5 + \;.\;.\;.\; + l2n$$
$$-2l1 - 2l2 - 2l3 - 2l4 - 2l5 \;.\;.\;.\; - 2ln$$

161. Iam vero ſumendis his logarithmis hyperbolicis erit:

$$l1 + l2 + l3 + l4 + \;.\;.\;.\;.\;.\; + l2n =$$
$$= \tfrac{1}{2}l2\pi + (2n + \tfrac{1}{2})ln + (2n + \tfrac{1}{2})l2 - 2n$$
$$+ \frac{\mathfrak{A}}{1.2.2n} - \frac{\mathfrak{B}}{3.4.2^3n^3} + \frac{\mathfrak{C}}{5.6.2^5n^5} - \&c.$$

& $2l1 + 2l2 + 2l3 + 2l4 + \;.\;.\;.\; + 2ln =$

$$l2\pi + (2n+1)ln - 2n + \frac{2\mathfrak{A}}{1.2n} - \frac{2\mathfrak{B}}{3.4n^3} + \frac{2\mathfrak{C}}{5.6n^5} - \&c.$$

qua expreſſione ab illa ſublata relinquetur:

$$lu = -\tfrac{1}{2}l\pi - \tfrac{1}{2}ln + 2nl2 + \frac{\mathfrak{A}}{1.2.2n} - \frac{\mathfrak{B}}{3.4.2^3n^3} + \frac{\mathfrak{C}}{5.6.2^5n^5} - \&c.$$
$$- \frac{2\mathfrak{A}}{1.2n} + \frac{2\mathfrak{B}}{3.4n^3} - \frac{2\mathfrak{C}}{5.6n^5} + \&c.$$

his vero binis terminis colligendis, erit:

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1.2.2n} + \frac{15\mathfrak{B}}{3.4.2^3n^3} - \frac{63\mathfrak{C}}{5.6.2^5n^5} + \frac{255\mathfrak{D}}{7.8.2^7n^7} - \&c.$$

Sit

Sit $\frac{3\mathfrak{A}}{1.2.2^2n^2} - \frac{15\mathfrak{B}}{3.4.2^4n^4} + \frac{63\mathfrak{C}}{5.6.2^6n^6} - \frac{255\mathfrak{D}}{7.8.2^8n^8} + \&c.$ $= l\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \frac{D}{2^8n^8} + \&c.\right)$; erit

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - 2nl\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \&c.\right);$$

ideoque $u = \dfrac{2^{2n}}{\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \&c.\right)^{2n}\sqrt{n\pi}}$

Erit vero poſito $2n = m$

$$l\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \frac{D}{2^8n^8} + \&c.\right) =$$

$$\begin{array}{l} \frac{A}{m^2} + \frac{B}{m^4} + \frac{C}{m^6} + \frac{D}{m^8} + \frac{E}{m^{10}} + \&c. \\ \qquad - \frac{A^2}{2m^4} - \frac{AB}{m^6} - \frac{AC}{m^8} - \frac{AD}{m^{10}} - \&c. \\ \qquad\qquad\qquad - \frac{BB}{2m^8} - \frac{BC}{m^{10}} - \&c. \\ \qquad\qquad + \frac{A^3}{3m^6} + \frac{A^2B}{m^8} + \frac{A^2C}{m^{10}} + \&c. \\ \qquad\qquad\qquad\qquad + \frac{AB^2}{m^{10}} + \&c. \\ \qquad\qquad\qquad - \frac{A^4}{4m^8} - \frac{A^3B}{m^{10}} - \&c. \\ \qquad\qquad\qquad\qquad + \frac{A^5}{5m^{10}} + \&c. \end{array}$$

quae expreſſio cum aequalis eſſe debeat huic:

$$\frac{3\mathfrak{A}}{1.2m^2} - \frac{15\mathfrak{B}}{3.4m^4} + \frac{63\mathfrak{C}}{5.6m^6} - \frac{255\mathfrak{D}}{7.8m^8} + \&c. \quad \text{fiet:}$$

A

$$A = \frac{3\mathfrak{A}}{1.2}$$

$$B = \frac{A^2}{2} - \frac{15\mathfrak{B}}{3.4}$$

$$C = AB - \tfrac{1}{3}A^3 + \frac{63\mathfrak{C}}{5.6}$$

$$D = AC + \tfrac{1}{2}B^2 - A^2B + \tfrac{1}{4}A^4 - \frac{255\mathfrak{D}}{7.8}$$

$$E = AD + BC - A^2C - AB^2 + A^3B - \tfrac{1}{5}A^5 + \frac{1023\mathfrak{E}}{9.10}$$

&c.

162. Cum iam ſit $\mathfrak{A} = \frac{1}{6}$; $\mathfrak{B} = \frac{1}{30}$; $\mathfrak{C} = \frac{1}{42}$; $\mathfrak{D} = \frac{1}{30}$; $\mathfrak{E} = \frac{5}{66}$; erit:

$$A = \frac{1}{4};\; B = -\frac{1}{96};\; C = \frac{27}{640};\; D = -\frac{90031}{2^{11}.3^2.5.7}\ \text{\&c.}$$

Hinc efficitur:

$$u = \frac{2^{2n}}{\left(1 + \frac{1}{2^4 n^2} - \frac{1}{2^9.3n^4} + \frac{27}{2^{13}.5n^6} - \frac{90031}{2^{19}.3^2.5.7n^8} + \text{\&c.}\right)^{2n}\sqrt{n\pi}}$$

vel ſi iſta ſeriei elevatio actu inſtituatur, erit proxime:

$$u = \frac{2^{2n}}{\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16.128n^4}\ \text{\&c.}\right)}$$

hinc terminus medius in $(1+1)^{2n}$, erit ad ſummam omnium terminorum 2^{2n}

uti 1 ad $\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16.128n^4} - \text{\&c.}\right)$

vel poſito brevitatis gratia $4n = \nu$, erit iſta ratio:

ut 1 ad $\sqrt{n\pi}\left(1+\frac{1}{\nu}+\frac{1}{2\nu^2}-\frac{1}{2\nu^3}-\frac{5}{8\nu^4}+\frac{23}{8\nu^5}+\frac{53}{16\nu^6}-\&c.\right)$

EXEMPLUM I.

Quaeratur uncia media in binomio $(a+b)^{10}$ *evoluto, quam conſtat eſſe* $=\frac{10.9.8.7.6}{1.2.3.4.5}=252.$

Adhibendo ultimam formulam pro u inventam erit $n=5$

$$\frac{1}{4n}=0,0500000$$

$$\frac{1}{32n^2}=0,0012500$$

$$0,0512500$$

ſubtr. $\frac{1}{128n^3}=625$

$$0,0511875$$

ſubtr. $\frac{5}{16.128n^4}=39$

Ergo $1+\frac{1}{4n}+\&c.=1,0511836$

Huius log. $=0,0216784$

$ln=0,6989700$

$l\pi=0,4971498$

$$1,2177982$$

$l\sqrt{n\pi}(1+\&c.)=0,6088991$

a $l2^{10}=3,0102999$

$lu=2,4014008$

unde fit $u=252.$

EXEMPLUM II.

Inveſtigetur ratio, quam in poteſtate centeſima binomii $1+1$ *terminus medius ad ſummam omnium* 2^{100} *tenet.*

Utamur ad hoc formula primum inventa:

$$lu=l\frac{2^{2n}}{\sqrt{n\pi}}-\frac{3\mathfrak{A}}{1.2.2n}+\frac{15\mathfrak{B}}{3.4.2^3n^3}-\frac{63\mathfrak{C}}{5.6.2^5n^5}+\&c.$$

in

in qua poſito $2n = m$ ut habeatur iſta poteſtas $(1+1)^m$ & loco 𝔄, 𝔅, ℭ, 𝔇, ſubſtitutis valoribus, fiet:

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \&c.$$

qui logarithmi cum ſint hyperbolici multiplicentur ii per

$$k = 0,434294481903251,$$

ut tranſmutentur in tabulares, eritque

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \&c.$$

unde cum uncia media ſit u, erit $2^m : u$ ratio quaeſita, ideoque

$$l\frac{2^m}{u} = l\sqrt{\tfrac{1}{2}m\pi} + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{112m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \&c.$$

Quare cum ſit ob exponentem $m = 100$

$$\frac{k}{m} = 0,0043429448; \quad \frac{k}{m^3} = 0,0000004343;$$

$$\frac{k}{m^5} = 0,0000000000;$$ erit:

$$\frac{k}{4m} = 0,0010857362$$

$$\frac{k}{24m^3} = 0,0000000181$$

$$0,0010857181$$

Tum eſt

$$l\pi = 0,4971498726$$

$$l\tfrac{1}{2}m = 1,6989700043$$

$$l\tfrac{1}{2}m\pi = 2,1961198769$$

$$l\sqrt{\tfrac{1}{2}m\pi} = 1,0980599384$$

$$\frac{k}{4m} - \frac{k}{24m^3} + \&c. = 0,0010857181$$

$$1,0991456565 = l\frac{2^{100}}{u}.$$

Erit ergo $\frac{2^{100}}{u} = 12,\ 56451$, atque adeo in poteſtate

$(1+1)^{100}$ evoluta terminus medius ſe habebit ad ſummam omnium 2^{100} uti 1 ad 12, 56451.

163. Denotet nunc terminus generalis z functionem exponentialem a^x, ita ut ſummari debeat haec ſeries geometrica:

$$s = a + a^2 + a^3 + a^4 + \ldots \ldots + a^x$$

quae cum ſit geometrica, eius ſumma iam conſtat, erit enim $s = \frac{(a^x - 1)a}{a-1}$. Modo autem hic expoſito hanc ſummam inveſtigemus. Quia eſt $z = a^x$, erit $\int z dx = \frac{a^x}{la}$, huius enim differentiale eſt $a^x dx$, tum vero erit:

$$\frac{dz}{dx} = a^x la;\ \frac{ddz}{dx^2} = a^x (la)^2;\ \frac{d^3z}{dx^3} = a^x (la)^3;\ \text{\&c.}$$

unde ſequitur fore:

$$s = a^x\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2}la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}}{1.2.3\ldots6}(la)^5 - \text{\&c.}\right) + C.$$

Ad conſtantem C definiendam ponatur $x = 0$, & ob $s = 0$, erit

$$C = -\frac{1}{la} - \frac{1}{2} - \frac{\mathfrak{A}}{1.2}la + \frac{\mathfrak{B}}{1.2.3.4}(la)^3 - \text{\&c.}$$

ideoque fiet:

$$s = (a^x - 1)\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2}la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}}{1.2.3\ldots6}(la)^5 - \text{\&c.}\right)$$

Cum igitur ſumma ſit $\frac{(a^x - 1)a}{a-1}$, erit:

$$\frac{a}{a-1} = \frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2}la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}(la)^5}{1.2\ldots6} - \text{\&c.}$$

ubi la denotat logarithmum hyperbolicum ipſius a: hinc fit

$$\frac{(a+1)la}{2(a-1)} = 1 + \frac{\mathfrak{A}(la)^2}{1.2} - \frac{\mathfrak{B}(la)^4}{1.2.3.4} + \frac{\mathfrak{C}(la)^6}{1.2\ldots6} - \text{\&c.}$$

ſicque iſtius ſeriei ſumma exhiberi poterit.

164. Sit terminus generalis $z = \text{ſin}\, a x$, &
$s = \text{ſin}\, a + \text{ſin}\, 2a + \text{ſin}\, 3a + \;.\;.\;.\;.\; + \text{ſin}\, ax$
quae ſeries cum ſit recurrens quoque ſummari poteſt; erit enim

$$s = \frac{\text{ſin}\, a + \text{ſin}\, ax - \text{ſin}\,(ax+a)}{1 - 2\,\text{coſ}\, a + 1} = \frac{\text{ſin}\, a + (1-\text{coſ}\, a)\text{ſin}\, ax - \text{ſin}\, a\,\text{coſ}\, ax}{2(1-\text{coſ}\, a)}$$

Erit vero $\int z dx = \int dx\, \text{ſin}\, ax = -\frac{1}{a}\text{coſ}\, ax$, & $\frac{dz}{dx} = a\,\text{coſ}\, ax$;

$\frac{ddz}{dx^2} = -aa\,\text{ſin}\, ax$; $\frac{d^3z}{dx^3} = -a^3\,\text{coſ}\, ax$; $\frac{d^5z}{dx^5} = a^5\,\text{coſ}\, ax$ &c.

$$s = C - \frac{1}{a}\text{coſ}\, ax + \tfrac{1}{2}\,\text{ſin}\, ax + \frac{\mathfrak{A}\, a\,\text{coſ}\, ax}{1.2} + \frac{\mathfrak{B}\, a^3\,\text{coſ}\, ax}{1.2.3.4} + \frac{\mathfrak{C}\, a^5\,\text{coſ}\, ax}{1.2.3.4.5.6} + \frac{\mathfrak{D}\, a^7\,\text{coſ}\, ax}{1.2\;.\;.\;.\;8} + \text{\&c.}$$

Ponatur $x = 0$ ut fiat $s = 0$; eritque:

$$C = \frac{1}{a} - \frac{\mathfrak{A}\, a}{1.2} - \frac{\mathfrak{B}\, a^3}{1.2.3.4} - \frac{\mathfrak{C}\, a^5}{1.2\ldots 6} - \text{\&c.}$$ ergo

$$s = \tfrac{1}{2}\,\text{ſin}\, ax + (1-\text{coſ}\, ax)\left(\frac{1}{a} - \frac{\mathfrak{A}a}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3.4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \text{\&c.}\right)$$

At cum ſit $s = \tfrac{1}{2}\,\text{ſin}\, ax + \frac{(1-\text{coſ}\, ax)\,\text{ſin}\, a}{2(1-\text{coſ}\, a)}$, fiet:

$$\frac{\text{ſin}\, a}{2(1-\text{coſ}\, a)} = \tfrac{1}{2}\cot\tfrac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3.4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \text{\&c.}$$

quam eandem ſeriem iam ſupra §. 127. habuimus.

165. Sit nunc $z = \text{coſ}\, ax$, ac ſeries ſummanda:
$s = \text{coſ}\, a + \text{coſ}\, 2a + \text{coſ}\, 3a + \;.\;.\;.\;.\;. + \text{coſ}\, ax$
cuius ſeriei, quia eſt recurrens, erit ſumma:

$$s = \frac{\text{coſ}\, a - 1 + \text{coſ}\, ax - \text{coſ}\,(ax+a)}{1 - 2\,\text{coſ}\, a + 1} = -\tfrac{1}{2} + \tfrac{1}{2}\,\text{coſ}\, ax + \tfrac{1}{2}\cot\tfrac{1}{2}a\,.\,\text{ſin}\, ax.$$

At vero ad ſummam noſtra methodo exprimendam, erit:

$\int z dx$

$$\int z dx = \int dx \operatorname{cof} ax = \frac{1}{a} \operatorname{fin} ax, \quad \& \quad \frac{dz}{dx} = -a \operatorname{fin} ax;$$

$$\frac{d^3 z}{dx^3} = a^3 \operatorname{fin} ax; \quad \frac{d^5 z}{dx^5} = -a^5 \operatorname{fin} ax; \quad \&c. \quad \text{Ergo}$$

$$s = C + \frac{1}{a} \operatorname{fin} ax + \tfrac{1}{2} \operatorname{cof} ax - \frac{\mathfrak{A}\, a \operatorname{fin} ax}{1.\,2} - \frac{\mathfrak{B}\, a^3 \operatorname{fin} ax}{1.\,2.\,3.\,4} - \&c.$$

Sit $x = 0$, erit $s = 0$, & $C = -\frac{1}{2}$, hincque erit:

$$s = -\tfrac{1}{2} + \tfrac{1}{2} \operatorname{cof} ax + \frac{1}{a} \operatorname{fin} ax - \frac{\mathfrak{A}\, a \operatorname{fin} ax}{1.\,2} - \frac{\mathfrak{B}\, a^3 \operatorname{fin} ax}{1.\,2.\,3.\,4} - \&c.$$

Quare cum ſit $s = -\frac{1}{2} + \frac{1}{2} \operatorname{cof} ax + \frac{1}{2} \cot \frac{1}{2} a . \operatorname{fin} ax$, erit uti iam modo invenimus:

$$\tfrac{1}{2} \cot \tfrac{1}{2} a = \frac{1}{a} - \frac{\mathfrak{A} a}{1.2} - \frac{\mathfrak{B} a^3}{1.\,2.\,3.\,4} - \frac{\mathfrak{C} a^5}{1.2.3.4.5.6} - \&c.$$

166. Quoniam ſupra invenimus, ſi a denotet arcum quemcunque, eſſe

$$\frac{\pi}{2} = \frac{a}{2} + \operatorname{fin} a + \tfrac{1}{2} \operatorname{fin} 2a + \tfrac{1}{3} \operatorname{fin} 3a + \tfrac{1}{4} \operatorname{fin} 4a + \&c.$$

conſideremus hanc ſeriem, ſitque $z = \frac{1}{x} \operatorname{fin} ax$, ut ſit

$$s = \operatorname{fin} a + \tfrac{1}{2} \operatorname{fin} 2a + \tfrac{1}{3} \operatorname{fin} 3a + \quad \ldots \quad + \frac{1}{x} \operatorname{fin} ax.$$

Hoc autem caſu ſit $\int z dx = \int \frac{dx}{x} \operatorname{fin} ax$, quod integrale exhiberi nequit.

Erit vero $\frac{dz}{dx} = \frac{a}{x} \operatorname{cof} ax - \frac{1}{xx} \operatorname{fin} ax$;

$$\frac{ddz}{dx^2} = -\frac{a^2}{x} \operatorname{fin} ax - \frac{2a}{xx} \operatorname{cof} ax + \frac{2}{x^3} \operatorname{fin} ax;$$

$$\frac{d^3 z}{dx^3} = -\frac{a^3}{x}\,\text{coſ}\,ax + \frac{3a^2}{x^2}\,\text{ſin}\,ax + \frac{6a}{x^3}\,\text{coſ}\,ax - \frac{6}{x^4}\,\text{ſin}\,ax;$$

$$\frac{d^4 z}{dx^4} = \frac{a^4}{x}\,\text{ſin}\,ax + \frac{4a^3}{xx}\,\text{coſ}\,ax - \frac{12a^2}{x^3}\,\text{ſin}\,ax - \frac{24a}{x^4}\,\text{coſ}\,ax + \frac{24}{x^5}\,\text{ſin}\,ax.$$

Quia igitur neque formulam integralem $\int z dx$ exhibere , neque haec differentialia ſatis commode exprimere licet , ſummam huius ſeriei per hanc methodum definire non poſſumus , ita ut quicquam inde concludi poſſet. Idem incommodum in multis aliis ſeriebus occurrit , quoties terminus generalis non ſatis eſt ſimplex , ut eius differentialia ad commodam legem exprimi queant . Quamobrem in ſequenti Capite alias expreſſiones generales pro ſummis ſerierum , quarum termini generales vel nimis ſunt compoſiti vel prorſus dari nequeunt , eliciemus ; quae feliciori ſucceſſu in uſum vocari poterunt. Imprimis autem inſufficientia methodi hic traditae elucet , ſi ſigna terminorum ſeriei propoſitae alternentur , tum enim quantumvis termini generales ſint ſimplices , tamen termini ſummatorii hac methodo exhiberi commode nequeunt.

CA-

CAPUT VII.

METHODUS SUMMANDI SUPERIOR ULTERIUS PROMOTA.

167.

Ut defectum methodi ſummandi ante traditae ſuppleamus, in hoc Capite eiuſmodi ſeries conſiderabimus, quarum termini generales magis ſint complexi. Cum igitur expreſſio ante inventa in progreſſionibus geometricis, etſi aliis methodis facillime ſummari poſſunt, veram ſummam finita formula contentam non praebeat, hic primum eiuſmodi ſeries contemplabimur, quarum termini ſint producta ex terminis feriei geometricae & alius cuiuſcunque. Sit igitur propoſita haec ſeries:

$$\overset{1}{s = ap} + \overset{2}{bp^2} + \overset{3}{cp^3} + \overset{4}{dp^4} + \ldots + \overset{x}{yp^x}$$

quae eſt compoſita ex geometrica p, p^2, p^3, &c. & alia quacunque ſerie $a + b + c + d +$ &c. cuius terminus generalis ſeu indici x reſpondens ſit $= y$, atque expreſſionem generalem inveſtigemus pro valore eius ſummae $s = S.yp^x$.

168. Inſtituamus ratiocinium eodem modo, quo ſupra uſi ſumus, ſitque v terminus antecedens ipſi y in ſerie $a + b + c + d +$ &c. atque A praecedens ipſi a ſeu is qui indici o reſpondet, eritque vp^{x-1} terminus generalis huius ſeriei:

$$\overset{1}{A} + \overset{2}{ap} + \overset{3}{bp^2} + \overset{4}{cp^3} + \ldots + \overset{x}{vp^{x-1}}$$

cuius ſumma, ſi indicetur per $S.vp^{x-1}$ erit:

$$S.vp^{x-1} = \frac{1}{p} S.vp^x = S.yp^x - yp^x + A.$$

Cum autem ſit:

$v =$

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \&c.$$

erit:

$$S.yp^x - yp^x + A = \frac{1}{p} S.yp^x - \frac{1}{p} S\frac{dy}{dx}p^x + \frac{1}{2p} S\frac{ddy}{dx^2}p^x - \frac{1}{6p} S\frac{d^3y}{dx^3}p^x + \frac{1}{24p} S\frac{d^4y}{dx^4}p^x - \&c.$$

Ex qua fit:

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S\frac{dy}{dx}p^x + S\frac{ddy}{2dx^2}p^x - S\frac{d^3y}{6dx^3}p^x + \&c.\right)$$

Si ergo habeantur termini ſummatorii ſerierum, quarum termini generales ſunt $\frac{dy}{dx}p^x$; $\frac{ddy}{dx^2}p^x$; $\frac{d^3y}{dx^3}p^x$; &c. ex iis definiri poterit terminus ſummatorius Syp^x.

169. Hinc iam ſummae inveniri poterunt ſerierum, quarum termini generales in hac forma $x^n p^x$ continentur. Sit enim $y = x^n$, erit $A = 0$, niſi ſit $n = 0$, quo caſu foret $A = 1$, & quia eſt:

$$\frac{dy}{dx} = nx^{n-1}; \frac{ddy}{2dx^2} = \frac{n(n-1)}{1.\ 2}x^{n-2}; \frac{d^3y}{dx^3} = \frac{n(n-1)(n-2)}{1.\ 2.\ 3}x^{n-3};$$

erit:

$$S.x^n p^x = \frac{1}{p-1}\left(x^n p^{x+1} - Ap - nS.x^{n-1}p^x + \frac{n(n-1)}{1.\ 2}S.x^{n-2}p^x - \frac{n(n-1)(n-2)}{1.\ 2.\ 3}S.x^{n-3}p^x + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}S.x^{n-4}p^x - \&c.\right)$$

Ex hac forma nunc ſucceſſive pro n ſubſtituendo numeros 0, 1, 2, 3, &c. obtinebuntur ſequentes ſummationes; ac primo quidem ſi $n = 0$, fit $A = 1$, in reliquis autem caſibus erit $A = 0$:

$$S.x^0 p^x = S.p^x = \frac{1}{p-1}(p^{x+1} - p) = \frac{p^{x+1} - p}{p-1} = \frac{p(p^x - 1)}{p-1},$$

quae eſt ſumma progreſſionis geometricae cognita:

$$S.xp^x = \frac{1}{p-1}(xp^{x+1} - S.p^x) = \frac{xp^{x+1}}{p-1} - \frac{p^{x+1}-p}{(p-1)^2}$$

ſeu $S.xp^x = \frac{pxp^x}{p-1} - \frac{p(p^x-1)}{(p-1)^2}$,

$$S.x^2p^x = \frac{1}{p-1}(x^2p^{x+1} - 2S.xp^x + S.p^x)$$ ſeu

$$S.x^2p^x = \frac{x^2p^{x+1}}{p-1} - \frac{2xp^{x+1}}{(p-1)^2} + \frac{p(p+1)(p^x-1)}{(p-1)^3},$$

Porro eſt

$$S.x^3p^x = \frac{1}{p-1}(x^3p^{x+1} - S.x^2p^x + 3S.xp^x - S.p^x)$$ ſeu

$$S.x^3p^x = \frac{x^3p^{x+1}}{p-1} - \frac{3x^2p^{x+1}}{(p-1)^2} + \frac{3(p+1)xp^{x+1}}{(p-1)^3} - \frac{p(pp+4p+1)(p^x-1)}{(p-1)^4}$$

ſicque ulterius progrediendo ſuperiorum poteſtatum x^4p^x; x^5p^x; x^6p^x; &c. ſummae definiri poterunt, hoc vero commodius praeſtabitur ope expreſſionis generalis, quam nunc inveſtigabimus.

170. Quoniam invenimus eſſe:

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3}p^x + \&c.\right)$$

ubi A eſt eiuſmodi conſtans, ut ſumma fiat $= 0$, ſi ponatur $x = 0$: namque hoc caſu fit $y = A$, & $yp^{x+1} = Ap$; hanc conſtantem omittere poterimus, dummodo perpetuo meminerimus ad ſummam quamque ſemper eiuſmodi conſtantem adiici oportere, ut facto $x = 0$, evaneſcat, ſeu ut alii cuipiam caſui ſatisfiat. Statuamus ergo z loco y, eritque

$$S.p^xz = \frac{p^{x+1}z}{p-1} - \frac{1}{p-1}S.p^x\frac{dz}{dx} + \frac{1}{2(p-1)}S.p^x\frac{ddz}{dx^2}$$

$$-\frac{1}{6(p-1)}S.p^x\frac{d^3z}{dx^3} + \frac{1}{24(p-1)}S.p^x\frac{d^4z}{dx^4} - \frac{1}{120(p-1)}S.p^x\frac{d^5z}{dx^5}$$

$$+ \&c.$$

Dein-

Deinde ſtatuamus ſucceſſive $\frac{dz}{dx}$; $\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. in locum y eritque:

$$S.\frac{p^x dz}{dx} = \frac{p^{x+1}}{p-1}.\frac{dz}{dx} - \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} + \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} - \&c.$$

$$S.\frac{p^x ddz}{dx^2} = \frac{p^{x+1}}{p-1}.\frac{ddz}{dx^2} - \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} + \frac{1}{2(p-1)}S.\frac{p^x d^4z}{dx^4} - \&c.$$

$$S.\frac{p^x d^3z}{dx^3} = \frac{p^{x+1}}{p-1}.\frac{d^3z}{dx^3} - \frac{1}{p-1}S.\frac{p^x d^4z}{dx^4} + \frac{1}{2(p-1)}S.\frac{p^x d^5z}{dx^5} - \&c.$$

&c.

Si igitur hi valores ſucceſſive ſubſtituantur, $S.p^x z$ huiuſmodi forma exprimetur:

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{\alpha p^{x+1}}{p-1}.\frac{dz}{dx} + \frac{\beta p^{x+1}}{p-1}.\frac{ddz}{dx^2} - \frac{\gamma p^{x+1}}{p-1}.\frac{d^3z}{dx^3} + \frac{\delta p^{x+1}}{p-1}.\frac{d^4z}{dx^4} - \frac{\varepsilon p^{x+1}}{p-1}.\frac{d^5z}{dx^5} + \&c.$$

171. Ad valores litterarum α, β, γ, δ, ε, &c. definiendos, ſubſtituantur pro quovis termino ſeries ante inventae nempe:

$$\frac{p^{x+1}z}{p-1} = S.p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} - \&c.$$

$$\frac{p^{x+1}dz}{(p-1)dx} = S.\frac{p^x dz}{dx} + \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} - \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} + \&c.$$

$$\frac{p^{x+1}ddz}{(p-1)dx^2} = S.\frac{p^x ddz}{dx^2} + \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} - \&c.$$

$$\frac{p^{x+1}d^3z}{(p-1)dx^3} = S.\frac{p^x d^3z}{dx^3} + \&c.$$

Habebimus ergo:

$$\begin{array}{llllllllll}
\multicolumn{10}{l}{S.p^x z \;=} \\
S.p^x z + & \frac{1}{p-1} S.\frac{p^x dz}{dx} & - & \frac{1}{2(p-1)} S.\frac{p^x ddz}{dx^2} & + & \frac{1}{6(p-1)} S.\frac{p^x d^3 z}{dx^3} & - & \frac{1}{24(p-1)} S.\frac{p^x d^4 z}{dx^4} & + \;\&c. \\
 & -\alpha & - & \frac{\alpha}{p-1} & + & \frac{\alpha}{2(p-1)} & - & \frac{\alpha}{6(p-1)} & \\
 & & + & \beta & + & \frac{\beta}{p-1} & - & \frac{\beta}{2(p-1)} & \\
 & & & & - & \gamma & - & \frac{\gamma}{p-1} & \\
 & & & & & & + & \delta &
\end{array}$$

unde coefficientium α, β, γ, δ, &c. valores ſequentes obtinebuntur.

$$\alpha = \frac{1}{p-1}$$

$$\beta = \frac{1}{p-1}\left(\alpha + \frac{1}{2}\right)$$

$$\gamma = \frac{1}{p-1}\left(\beta + \frac{\alpha}{2} + \frac{1}{6}\right)$$

$$\delta = \frac{1}{p-1}\left(\gamma + \frac{\beta}{2} + \frac{\alpha}{6} + \frac{1}{24}\right)$$

$$\varepsilon = \frac{1}{p-1}\left(\delta + \frac{\gamma}{2} + \frac{\beta}{6} + \frac{\alpha}{24} + \frac{1}{120}\right) \quad \&c.$$

172. Sit brevitatis gratia $\frac{1}{p-1} = q$, erit:

$$\alpha = q$$

$$\beta = \alpha q + \tfrac{1}{2} q = qq + \tfrac{1}{2} q$$

$$\gamma = \beta q + \tfrac{1}{2}\alpha q + \tfrac{1}{6} q = q^3 + qq + \tfrac{1}{6} q$$

$$\delta = \gamma q + \tfrac{1}{2}\beta q + \tfrac{1}{6}\alpha q + \tfrac{1}{24} q = q^4 + \tfrac{3}{2} q^3 + \tfrac{7}{12} q^2 + \tfrac{1}{24} q$$

$$\varepsilon = \delta q + \tfrac{1}{2}\gamma q + \tfrac{1}{6}\beta q + \tfrac{1}{24}\alpha q + \tfrac{1}{120} q$$

ſeu $\varepsilon = q^5 + 2q^4 + \tfrac{5}{4} q^3 + \tfrac{1}{4} q^2 + \tfrac{1}{120} q$ &

$$\zeta = q^6 + \tfrac{5}{2} q^5 + \tfrac{13}{6} q^4 + \tfrac{3}{4} q^3 + \tfrac{31}{360} q^2 + \tfrac{1}{720} q \;\&c.$$

ſeu

ſeu hoc modo exprimantur:

$$\alpha = \frac{q}{1}$$
$$\beta = \frac{2qq + q}{1.\,2}$$
$$\gamma = \frac{6q^3 + 6q^2 + q}{1.\,2.\,3}$$
$$\delta = \frac{24q^4 + 36q^3 + 14q^2 + q}{1.\,2.\,3.\,4}$$
$$\varepsilon = \frac{120q^5 + 240q^4 + 150q^3 + 30q^2 + q}{1.\,2.\,3.\,4.\,5}$$
$$\zeta = \frac{720q^6 + 1800q^5 + 1560q^4 + 540q^3 + 62q^2 + q}{1.\,2.\,3.\,4.\,5.\,6}$$
$$\eta = \frac{5040q^7 + 15120q^6 + 16800q^5 + 8400q^4 + 1806q^3 + 126q^2 + q}{1.\,2.\,3.\,4.\,5.\,6.\,7} \quad \&c.$$

ubi quilibet coefficiens 16800 oritur, ſi ſumma binorum ſuperiorum $1560 + 1800$ per exponentem ipſius q, qui hic eſt 5, multiplicetur.

173. Reſtituamus autem loco q valorem $\frac{1}{p-1}$,

$$\alpha = \frac{1}{1(p-1)}$$
$$\beta = \frac{p+1}{1.2\,(p-1)^2}$$
$$\gamma = \frac{pp + 4p + 1}{1.2.3\,(p-1)^3}$$
$$\delta = \frac{p^3 + 11p^2 + 11p + 1}{1.\,2.\,3.\,4\,(p-1)^4}$$
$$\varepsilon = \frac{p^4 + 26p^3 + 66p^2 + 26p + 1}{1.\,2.\,3.\,4.\,5\,(p-1)^5}$$

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$$\zeta = \frac{p^5 + 57p^4 + 302p^3 + 302p^2 + 57p + 1}{1.\,2.\,3.\,4.\,5.\,6\,(p-1)^6}$$

$$\eta = \frac{p^6 + 120p^5 + 1191p^4 + 2416p^3 + 1191p^2 + 120p + 1}{1.\,2.\,3.\,4.\,5.\,6.\,7\,(p-1)^7}$$

&c.

Lex harum quantitatum ita ſe habet, ut ſi ponatur terminus quicunque:

$$\frac{p^{n-2} + Ap^{n-3} + Bp^{n-4} + Cp^{n-5} + Dp^{n-6} + \&c.}{1.\,2.\,3\;.\;.\;.\;.\;.\;.\;(n-1)(p-1)^{n-1}}$$

futurum ſit:

$$A = 2^{n-1} - n$$

$$B = 3^{n-1} - n.2^{n-1} + \frac{n(n-1)}{1.\,2}$$

$$C = 4^{n-1} - n.3^{n-1} + \frac{n(n-1)}{1.\,2}2^{n-1} - \frac{n(n-1)(n-2)}{1.\,2.\,3}$$

$$D = 5^{n-1} - n.4^{n-1} + \frac{n(n-1)}{1.\,2}3^{n-1} - \frac{n(n-1)(n-2)}{1.\,2.\,3}2^{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4}$$

&c.

unde iſti coefficientes α, β, γ, δ, &c. quouſque libuerit, continuari poſſunt.

174. Quodſi vero legem, qua hi coefficientes inter ſe cohaerent, conſideremus, facile patet, eos ſeriem recurrentem conſtituere, atque prodire ſi haec fractio evolvatur:

$$\frac{1}{1 - \frac{u}{p-1} - \frac{u^2}{2(p-1)} - \frac{u^3}{6(p-1)} - \frac{u^4}{24(p-1)} - \&c.}$$

prodibit enim haec ſeries:

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

Ponatur illa fractio $= V$, & cum ſit:

$$V = \frac{p-1}{p - 1 - u - \frac{u^2}{2} - \frac{u^3}{6} - \frac{u^4}{24} - \&c.}$$

erit $V = \frac{p - 1}{p - e^u}$; ubi e eſt numerus cuius logarithmus hyperbolicus eſt $= 1$.

Atque ſi valor ipſius V per ſeriem exprimatur ſecundum poteſtates ipſius u, orietur:

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

cuius coefficientes α, β, γ, δ, &c. erunt ii ipſi, quorum in praeſenti negotio opus habemus. Iis igitur inventis erit:

$$S.p^x z = \frac{p^{x+1}}{p-1}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right) \pm \text{Conſt.}$$

quae ergo expreſſio eſt terminus ſummatorius ſeriei huius:

$$ap + bp^2 + cp^3 + \quad . \quad . \quad . \quad . \quad . \quad . \quad + p^x z$$

cuius terminus generalis eſt $= p^x z$.

175. Quoniam invenimus eſſe $V = \frac{p-1}{p-e^u}$, erit $e^u = \frac{pV - p + 1}{V}$, & logarithmis ſumendis fiet $u = l(pV - p + 1) - lV$, hincque; differentiando $du = \frac{(p-1)dV}{pV^2 - (p-1)V}$ quocirca erit $pV^2 = (p-1)V + \frac{(p-1)dV}{du}$. Quoniam ergo eſt $V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \&c.$ erit:

$$\begin{aligned} pV^2 = p &+ 2\alpha pu + 2\beta pu^2 + 2\gamma pu^3 + 2\delta pu^4 + 2\varepsilon pu^5 + \&c. \\ &+ \alpha^2 pu^2 + 2\alpha\beta pu^3 + 2\alpha\gamma pu^4 + 2\alpha\delta pu^5 + \&c. \\ &+ \beta^2 pu^4 + 2\beta\gamma pu^5 + \&c. \end{aligned}$$

$$\begin{aligned} (p-1)V = p - 1 &+ \alpha(p-1)u + \beta(p-1)u^2 + \gamma(p-1)u^3 \\ &+ \delta(p-1)u^4 + \varepsilon(p-1)u^5 + \&c. \end{aligned}$$

$$\begin{aligned} \frac{(p-1)dV}{du} = (p-1)\alpha &+ 2(p-1)\beta u + 3(p-1)\gamma u^2 + 4(p-1)\delta u^3 \\ &+ 5(p-1)\varepsilon u^4 + 6(p-1)\zeta u^5 + \&c. \end{aligned}$$

quibus expreſſionibus inter ſe coaequatis reperietur:

(p

$$(p-1)\,\alpha = 1$$
$$2\,(p-1)\,\beta = \alpha\,(p+1)$$
$$3\,(p-1)\,\gamma = \beta\,(p+1) + \alpha^2 p$$
$$4\,(p-1)\,\delta = \gamma\,(p+1) + 2\alpha\beta p$$
$$5\,(p-1)\,\varepsilon = \delta\,(p+1) + 2\alpha\gamma p + \beta^2 p$$
$$6\,(p-1)\,\zeta = \varepsilon\,(p+1) + 2\alpha\delta p + 2\beta\gamma p$$
$$7\,(p-1)\,\eta = \zeta\,(p+1) + 2\alpha\varepsilon p + 2\,\beta\delta p + \gamma^2 p$$
&c.

ex quibus formulis, ſi pro p datus numerus aſſumatur, valores coefficientium α, β, γ, δ, &c. facilius determinari poſſunt, quam ex lege primum inventa.

176. Antequam ad caſus ſpeciales ratione valoris ipſius p deſcendamus, ponamus eſſe $z = x^n$, ita ut haec ſeries ſummari debeat:

$$s = p + 2^n p^2 + 3^n p^3 + 4^n p^4 + \ \cdot\ \cdot\ \cdot\ + x^n p^x$$

eritque per expreſſionem ante inventam:

$$s = p^x\left(\frac{p}{p-1}\cdot x^n - \frac{p}{(p-1)^2}nx^{n-1} + \frac{pp+p}{(p-1)^3}\cdot\frac{n(n-1)}{1.\ 2}x^{n-2} - \frac{(p^3+4p^2+p)}{(p-1)^4}\cdot\frac{n(n-1)(n-2)}{1.\ 2.\ 3}x^{n-3} + \&c.\right)$$

$\pm$ C, quae reddat $s = 0$ ſi ponatur $x = 0$.
Hinc ponendo pro n ſucceſſive numeros 0, 1, 2, 3, 4, &c. erit:

$$S.x^0 p^x = p^x\cdot\frac{p}{p-1} - \frac{p}{p-1}$$

$$S.x^1 p^x = p^x\left(\frac{px}{p-1} - \frac{p}{(p-1)^2}\right) + \frac{p}{(p-1)^2}$$

$$S.x^2 p^x = p^x\left(\frac{px^2}{p-1} - \frac{2px}{(p-1)^2} + \frac{p(p+1)}{(p-1)^3}\right) - \frac{p(p+1)}{(p-1)^3}$$

$$S.x^3 p^x = p^x\left(\frac{px^3}{p-1} - \frac{3px^2}{(p-1)^2} + \frac{3p(p+1)x}{(p-1)^3} - \frac{p(p^2+4p+1)}{(p-1)^4}\right) + \frac{p(p^2+4p+1)}{(p-1)^4}$$

S.x^4

$$S.x^4p^x = p^x\left(\frac{px^4}{p-1} - \frac{4px^3}{(p-1)^2} + \frac{6p(p+1)x^2}{(p-1)^3} - \frac{4p(p^2+4p+1)x}{(p-1)^4}\right.$$
$$\left. + \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}\right) - \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}$$

$$S.x^5p^x = \frac{p^{x+1}x^5}{p-1} - \frac{5p^{x+1}x^4}{(p-1)^2} + \frac{10(p+1)p^{x+1}x^3}{(p-1)^3}$$
$$- \frac{10(p^2+4p+1)p^{x+1}x^2}{(p-1)^4} + \frac{5(p^3+11p^2+11p+1)p^{x+1}x}{(p-1)^5}$$
$$- \frac{(p^4+26p^3+66p^2+26p+1)(p^{x+1}-p)}{(p-1)^6}$$

$$S.x^6p^x = \frac{p^{x+1}x^6}{p-1} - \frac{6p^{x+1}x^5}{(p-1)^2} + \frac{15(p+1)p^{x+1}x^4}{(p-1)^3}$$
$$- \frac{20(p^2+4p+1)p^{x+1}x^3}{(p-1)^4} + \frac{15(p^3+11p^4+11p+1)p^{x}+1x^2}{(p-1)^5}$$
$$- \frac{6(p^4+26p^3+66p^4+26p+1)p^{x+1}x}{(p-1)^6}$$
$$+ \frac{(p^5+57p^4+302p^3+302p^2+57p+1)(p^{x+1}-p)}{(p-1)^7} \quad \&c.$$

177. Hinc intelligitur, quoties z fuerit functio rationalis integra ipſius x, toties ſeriei, cuius terminus generalis eſt p^xz, ſummam exhiberi poſſe; propterea quod differentialia ipſius z ſumendo, tandem ad evaneſcentia perveniatur. Ita ſi proponatur haec ſeries:

$$p + 3p^2 + 6p^3 + 10p^4 + \quad . \quad . \quad . \quad + \frac{(xx+x)}{2}p^x,$$

ob $z = \frac{xx+x}{2}$, & $\frac{dz}{dx} = x + \frac{1}{2}$; atque $\frac{ddz}{dx^2} = 1$;

erit terminus ſummatorius:

$$s = \frac{p^{x+1}}{p-1}\left(\tfrac{1}{2}xx + \tfrac{1}{2}x - \frac{2x+1}{2(p-1)} + \frac{p+1}{2(p-1)^2}\right) - \frac{p}{p-1}\left(\frac{p+1}{2(p-1)^2} - \frac{1}{2(p-1)}\right)$$

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ſeu $s = p^{x+1}\left(\frac{xx}{2(p-1)} + \frac{(p-3)x}{2(p-1)^2} + \frac{1}{(p-1)^3}\right) - \frac{p}{(p-1)^3}$.
Sin autem z fuerit functio non rationalis integra, tum iſta termini ſummatorii expreſſio in infinitum excurret. Ita ſi ſit $z = \frac{1}{x}$, ut ſummanda ſit haec ſeries:

$$s = p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \frac{1}{4}p^4 + \ldots\ldots + \frac{1}{x}p^x,$$ ob

$$\frac{dz}{dx} = -\frac{1}{xx};\ \frac{ddz}{dx^2} = \frac{2}{x^3};\ \frac{d^3z}{dx^3} = -\frac{2\cdot3}{x^4};\ \frac{d^4z}{dx^4} = \frac{2\cdot3\cdot4}{x^5};\ \&c.$$

prodibit terminus ſummatorius:

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{x} + \frac{1}{(p-1)x^2} + \frac{p+1}{(p-1)^2x^3} + \frac{pp+4p+1}{(p-1)^3x^4} + \frac{p^3+11p^2+11p+1}{(p-1)^4x^5} + \&c.\right) + C.$$

Hoc ergo caſu conſtans C non ex caſu $x = 0$ definiri poteſt: ad eam igitur definiendam ponatur $x = 1$, & quia fit $s = p$, erit:

$$C = p - \frac{pp}{p-1}\left(1 + \frac{1}{p-1} + \frac{p+1}{(p-1)^2} + \frac{pp+4p+1}{(p-1)^3} + \&c.\right)$$

178. Ex his perſpicuum eſt, niſi p determinatum numerum ſignificet, parum utilitatis hinc ad ſummas ſerierum proxime exhibendas redundare. Primum autem patet pro p non poſſe ſcribi 1, propterea quod omnes coefficientes α, β, γ, δ, &c. fierent infinite magni. Quare cum ſeries, quam nunc tractamus, abeat in eam quam ante iam ſumus contemplati ſi ponatur $p = 1$, mirum eſt, quod ille caſus tanquam facillimus ex hoc erui nequeat. Tum vero quoque notabile eſt, quod caſu $p = 1$ ſummatio requirat integrale $\int zdx$, cum tamen generaliter ſumma ſine ullo integrali exhiberi queat. Sic igitur fit, ut dum omnes coefficientes α, β, γ, δ, &c. in infinitum excreſcunt, ſimul formula illa integralis invehatur. Hicque adeo caſus, quo $p = 1$, eſt ſolus, ad

quem

quem generalis expreſſio hic inventa applicari nequeat. Neque vero hoc caſu generalis forma a vero recedere cenſenda eſt; nam etſi ſinguli termini fiunt infiniti, tamen revera omnia infinita ſe deſtruunt, reſtatque quantitas finita ſummae aequalis, & congruens cum ea, quae per priorem methodum invenitur, quod infra fuſius ſumus declaraturi.

179. Sit igitur $p = -1$, atque ſigna in ſerie ſummanda alternatim ſe excipient:

$$\overset{1}{-a} + \overset{2}{b} - \overset{3}{c} + \overset{4}{d} \; . \; . \; . \; . \; . \; \pm \overset{x}{z}$$

ubi z erit affirmativum ſi x fuerit numerus par, negativum autem, ſi x ſit numerus impar. Poſito ergo

$$-a + b - c + d \; . \; . \; . \; \pm z = s, \text{ erit}$$

$$s = \frac{\pm 1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right) + C.$$

ubi ſignorum ambiguorum ſuperius valet, ſi x ſit numerus par, contra vero ſi x ſit numerus impar. Mutandis ergo ſignis erit:

$$a - b + c - d + e - f + \; . \; . \; . \; \mp z =$$

$$\mp \frac{1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right) + C.$$

ubi ſignorum ambiguitas eandem ſequitur legem.

180. Hoc caſu coefficientes α, β, γ, δ, ε, ζ, &c. inveniri poſſunt ex valoribus ante traditis ponendo ubique $p = -1$. Facilius autem eruentur ex formulis generalibus §. 175. datis, ex quibus ſimul perſpicietur alternos iſtos coefficientes evaneſcere. Facto enim $p = -1$ iſtae formulae abibunt in

$$\begin{aligned}
-2\alpha &= 1 \\
-4\beta &= 0 \\
-6\gamma &= 0 - \alpha^2 \\
-8\delta &= 0 - 2\alpha\beta \\
-10\varepsilon &= 0 - 2\alpha\gamma - \beta\beta \\
-12\zeta &= 0 - 2\alpha\delta - 2\beta\gamma \qquad \&c.
\end{aligned}$$

 un-

unde cum ſit $\beta = 0$, erit quoque $\delta = 0$, porroque $\zeta = 0$, $\theta = 0$, &c. & reliquae litterae ita determinabuntur, ut ſit:

$$\alpha = -\frac{1}{2}\ ;\quad \gamma = \frac{\alpha^2}{6}\ ;\quad \varepsilon = \frac{2\alpha\gamma}{10}\ ;$$

$$\eta = \frac{2\alpha\varepsilon + \gamma\gamma}{14}\ ;\quad \iota = \frac{2\alpha\eta + 2\gamma\varepsilon}{18}\ ;\ \&c.$$

181. Quo iſte calculus commodius abſolvi poſſit introducamus novas litteras ſitque:

$$\alpha = -\frac{A}{1.2}\ ;\quad \gamma = \frac{B}{1.2.3.4}\ ;\quad \varepsilon = -\frac{C}{1.2.3.4.5.6}\ ;$$

$$\eta = \frac{D}{1.2.3\ldots.8}\ ;\quad \iota = -\frac{E}{1.2.3\ldots10}\ ;\quad \&c.$$

Eritque ſumma ante exhibita:

$$\mp\frac{1}{2}\left(z + \frac{Adz}{1.2dx} - \frac{Bd^3z}{1.2.3.4dx^3} + \frac{Cd^5z}{1.2\ldots6dx^5} - \frac{Dd^7z}{1.2\ldots8dx^7} + \&c.\right)$$
$$+\ \text{Conſt.}$$

Coefficientes vero ex ſequentibus formulis definientur:

$$A = 1$$

$$3B = \frac{4.3}{1.2}\,\frac{AA}{2}$$

$$5C = \frac{6.5}{1.2}\,AB$$

$$7D = \frac{8.7}{1.2}\,AC + \frac{8.7.6.5}{1.2.3.4}\cdot\frac{BB}{2}$$

$$9E = \frac{10.9}{1.2}\,AD + \frac{10.9.8.7}{1.2.3.4}\cdot BC$$

$$11F = \frac{12.11}{1.2.}\,AE + \frac{12.11.10.9}{1.2.3.4}\cdot BD + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}\cdot\frac{CC}{2}$$

&c.

quae hoc modo facilius atque ad calculum accommodatius repraeſentari poſſunt:

A

$$A = 1$$
$$B = 2.\frac{AA}{2}$$
$$C = 3.AB$$
$$D = 4.AC + 4.\frac{6.5}{3.4}.\frac{BB}{2}$$
$$E = 5.AD + 5.\frac{8.7}{3.4}.BC$$
$$F = 6.AE + 6.\frac{10.9}{3.4}.BD + 6.\frac{10.9.8.7}{3.4.5.6}.\frac{CC}{2}$$
$$G = 7.AF + 7.\frac{12.11}{3.4}.BE + 7.\frac{12.11.10.9}{3.4.5.6}.CD$$
&c.

Hinc igitur calculo inſtituto reperietur:

$A = 1$
$B = 1$
$C = 3$
$D = 17$
$E = 155 = 5.31$
$F = 2073 = 691.3$
$G = 38227 = 7.5461 = 7.\frac{127.129}{3}$
$H = 929569 = 3617.257$
$I = 28820619 = 43867.9.73$ &c.

182. Si hos numeros attentius perpendamus, ex factoribus 691, 3617, 43867, facile concludere licet, hos numeros cum ſupra exhibitis Bernoullianis nexum habere, indeque determinari poſſe. Hanc igitur relationem inveſtiganti mox patebit hos numeros ex Bernoullianis 𝔄, 𝔅, ℭ, 𝔇, 𝔈, &c. ſequenti modo formari poſſe:

A

$$
\begin{aligned}
A &= 2.\ 1.\ 3\ \mathfrak{A} = 2(2^2-1)\,\mathfrak{A}\\
B &= 2.\ 3.\ 5\ \mathfrak{B} = 2(2^4-1)\,\mathfrak{B}\\
C &= 2.\ 7.\ 9\ \mathfrak{C} = 2(2^6-1)\,\mathfrak{C}\\
D &= 2.\ 15.17\ \mathfrak{D} = 2(2^8-1)\,\mathfrak{D}\\
E &= 2.\ 31.33\ \mathfrak{E} = 2(2^{10}-1)\,\mathfrak{E}\\
F &= 2.\ 63.65\ \mathfrak{F} = 2(2^{12}-1)\,\mathfrak{F}\\
G &= 2.127.129\ \mathfrak{G} = 2(2^{14}-1)\,\mathfrak{G}\\
H &= 2.255.257\ \mathfrak{H} = 2(2^{16}-1)\,\mathfrak{H}\\
&\text{\&c.}
\end{aligned}
$$

Cum igitur numeri Bernoulliani ſint fracti, coefficientes vero noſtri integri, patet hos factores ſemper tollere fractiones; eruntque ergo:

$$
\begin{aligned}
A &= 1\\
B &= 1\\
C &= 3\\
D &= 17\\
E &= 5.31 = 155\\
F &= 3.691 = 2073\\
G &= 7.43.127 = 38227\\
H &= 257.3617 = 929569\\
I &= 9.73.43867 = 28820619\\
K &= 5.31.41.174611 = 1109652905\\
L &= 89.683.854513 = 51943281731\\
M &= 3.4097.236364091 = 2905151042481\\
N &= 2731.8191.8553103 = 191329672483963\\
&\text{\&c.}
\end{aligned}
$$

Ex his ergo numeris integris viciſſim numeri Bernoulliani inveniri poterunt.

183. Adhibendo igitur numeros Bernoullianos ſeriei propoſitae: $\overset{1}{a}-\overset{2}{b}+\overset{3}{c}-\overset{4}{d}+\overset{5}{e}-\ \ldots\ \mp\overset{x}{z}$, ſumma erit:

$$\mp\left(\frac{1}{2}z+\frac{(2^2-1)\mathfrak{A}dz}{1.2dx}-\frac{(2^4-1)\mathfrak{B}d^3z}{1.2.3.4dx^3}+\frac{(2^6-1)\mathfrak{C}d^5z}{1.2\ldots 6dx^5}-\frac{(2^8-1)\mathfrak{D}d^7z}{1.2\ldots 8dx^7}+\&c.\right)$$
$$+\ \text{Conſt.}$$

Hinc

Hinc autem perſpicitur iſtos numeros non caſu in hanc expreſſionem ingredi; quemadmodum enim ſeries propoſita oritur, ſi ab iſta: $a+b+c+d+ \ldots\ldots +z$, ubi omnes termini ſignum habent $+$ ſubtrahatur ſumma alternorum $b+d+f+$ &c. bis ſumta; ita quoque expreſſio inventa in duas reſolvi poteſt partes, quarum altera eſt ſumma omnium terminorum ſigno $+$ affectorum, quae erit:

$$\int z dx + \frac{1}{2}z + \frac{\mathfrak{A}dz}{1.2dx} - \frac{\mathfrak{B}d^3z}{1.2.3.4dx^3} + \frac{\mathfrak{C}d^5z}{1.2\ldots 6dx^5} - \&c.$$

Summa vero alternorum pari modo invenietur, quo ſupra uſi ſumus. Cum enim ultimus terminus ſit z indici x reſpondens, antecedens indici $x-2$ reſpondens erit;

$$z - \frac{2dz}{dx} + \frac{2^2ddz}{1.2dx^2} - \frac{2^3d^3z}{1.2.3dx^3} + \frac{2^4d^4z}{1.2.3.4dx^4} - \&c.$$

quae forma ex illa, qua ante terminus antecedens exprimebatur, oritur, ſi loco x ſcribatur $\frac{x}{2}$. Habebitur ergo ſumma alternorum, ſi in ſumma omnium ubique loco x ſcribatur $\frac{x}{2}$, quae propterea erit:

$$\frac{1}{2}\int z dx + \frac{1}{2}z + \frac{2\mathfrak{A}dz}{1.2dx} - \frac{2^3\mathfrak{B}d^3z}{1.2.3.4dx^3} + \frac{2^5\mathfrak{C}d^5z}{1.2\ldots 6dx^5} - \&c.$$

cuius duplum ſi a ſumma praecedente ſubtrahatur, exiſtente x numero pari, vel ſi praecedens ſumma a duplo huius ſi x eſt numerus impar ſubtrahatur, reſiduum oſtendet ſummam ſeriei:

$$\overset{1}{a} - \overset{2}{b} + \overset{3}{c} - \overset{4}{d} + \overset{5}{e} \ldots\ldots \mp \overset{x}{z}$$

quae ergo erit:

$$\mp\left(\frac{1}{2}z + \frac{(2^2-1)\mathfrak{A}dz}{1.2dx} - \frac{(2^4-1)\mathfrak{B}d^3z}{1.2.3.4dx^3} + \&c.\right) + C.$$

quae eſt eadem expreſſio, quam modo inveneramus.

184.

184. Sumatur pro z poteſtas ipſius x, nempe x^n, ut reperiatur ſumma ſeriei:

$$1 - 2^n + 3^n - 4^n + \quad . \quad . \quad . \quad . \quad \mp x^n.$$

Ob $\frac{dz}{1dx} = \frac{n}{1} x^{n-1}$; $\frac{d^3z}{1.2.3\,dx^3} = \frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3}$; &c. erit adhibendis coefficientibus A, B, C, &c. ſumma quaeſita:

$$\mp \frac{1}{2}\left(x^n + \frac{A}{2} nx^{n-1} - \frac{B}{4}\frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3} + \frac{C}{6}\frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5} x^{n-5} \right.$$
$$\left. - \frac{D}{8}\frac{n(n-1)\ldots\ldots(n-6)}{1.\ 2 \ldots\ldots 7} x^{n-7} + \&c.\right) + \text{Conſt.}$$

ubi ſignum ſuperius valet ſi ſit x numerus par, inferius vero ſi impar. Conſtans autem ita definiri debet, ut ſumma evaneſcat, ſi $x = 0$, quo caſu ſignum ſuperius valet. Pro n ergo ſucceſſive numeros 0, 1, 2, 3, &c. ſubſtituendo ſequentes prodibunt ſummationes:

I. $1 - 1 + 1 - 1 + \quad . \quad . \quad . \quad . \quad \mp 1 = \mp \frac{1}{2}(1) + \frac{1}{2}$
ſcilicet ſi numerus terminorum fuerit par, ſumma erit $= 0$, ſin impar erit $= +1$.

II. $1 - 2 + 3 - 4 + \quad . \quad . \quad . \quad \mp x = \mp \frac{1}{2}(x + \frac{1}{2}) + \frac{1}{4}$
ſcilicet ſi numerus terminorum ſit par, ſumma erit $= -\frac{1}{2}x$ & pro numero terminorum impari $= +\frac{1}{2}x + \frac{1}{2}$.

III. $1 - 2^2 + 3^2 - 4^2 + \quad . \quad . \quad \mp x^2 = \mp \frac{1}{2}(x^2 + x)$
ſcilicet pro pari numero $= -\frac{1}{2}xx - \frac{1}{2}x$
& pro impari numero $= +\frac{1}{2}xx + \frac{1}{2}x$

IV. $1 - 2^3 + 3^3 - 4^3 + \quad . \quad . \quad \mp x^3 = \mp \frac{1}{2}(x^3 + \frac{3}{2}xx - \frac{1}{4}) - \frac{1}{8}$
ſcilicet pro pari $= -\frac{1}{2}x^3 - \frac{3}{4}x^2$
& pro impari $= \frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{4}$.

V. $1 - 2^4 + 3^4 - 4^4 + \quad . \quad . \quad \mp x^4 = \mp \frac{1}{2}(x^4 + 2x^3 - x)$
ſcilicet pro numero pari $= -\frac{1}{2}x^4 - x^3 + \frac{1}{2}x$
& pro numero impari $= \frac{1}{2}x^4 + x^3 - \frac{1}{2}x$
&c.

185. Apparet ergo in poteſtatibus paribus praeter $n = 0$, conſtantem adiiciendam evaneſcere, hiſque caſibus ſum-

ſummam terminorum numero ſive parium ſive imparium tantum ratione ſigni diſcrepare. Quodſi ergo x fuerit numerus infinitus, quoniam is eſt neque par neque impar, haec conſideratio ceſſare debet, ac propterea in ſumma termini ambigui ſunt reiiciendi: unde ſequitur huiuſmodi ſerierum in infinitum continuatarum ſummam exprimi per ſolam quantitatem conſtantem adiiciendam. Hancobrem erit:

$$1 - 1 + 1 - 1 + \&c. \text{ in infinitum} = \frac{1}{2}$$

$$1 - 2 + 3 - 4 + \&c. \quad \ldots = \frac{A}{4} = + \frac{(2^2-1)\mathfrak{A}}{2}$$

$$1 - 2^2 + 3^2 - 4^2 + \&c. \quad \ldots = 0$$

$$1 - 2^3 + 3^3 - 4^3 + \&c. \quad \ldots = -\frac{B}{8} = -\frac{(2^4-1)\mathfrak{B}}{4}$$

$$1 - 2^4 + 3^4 - 4^4 + \&c. \quad \ldots = 0$$

$$1 - 2^5 + 3^5 - 4^5 + \&c. \quad \ldots = \frac{C}{12} = + \frac{(2^6-1)\mathfrak{C}}{6}$$

$$1 - 2^6 + 3^6 - 4^6 + \&c. \quad \ldots = 0$$

$$1 - 2^7 + 3^7 - 4^7 + \&c. \quad \ldots = -\frac{D}{16} = -\frac{(2^8-1)\mathfrak{D}}{8}$$

&c.

Quae eaedem ſummae per methodum ſupra traditam ſeries, in quibus ſigna + & — alternantur, ſummandi inveniuntur.

186. Si pro n ſtatuantur numeri negativi, expreſſio ſummae in infinitum excurret. Sit $n = -1$, erit ſumma ſeriei:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \mp \frac{1}{x} =$$

$$\mp \frac{1}{2}\left(\frac{1}{x} - \frac{A}{2x^2} + \frac{B}{4x^4} - \frac{C}{6x^6} + \frac{D}{8x^8} - \&c.\right) + \text{Conſt.}$$

Hic autem quia conſtans non ex caſu $x = 0$ definiri poteſt,

ex alio caſu erit definienda. Ponatur $x = 1$, atque ob ſummam $= 1$ & ſignum inferius erit:

$$\text{Conſt.} = 1 - \frac{1}{2}\left(\frac{1}{1} - \frac{A}{2} + \frac{B}{4} - \frac{C}{6} + \&c.\right) \quad \text{ſeu}$$

$$\text{Conſt.} = \frac{1}{2} + \frac{A}{4} - \frac{B}{8} + \frac{C}{12} - \frac{D}{16} + \&c.$$

Vel ponatur $x = 2$, ob ſummam $= \frac{1}{2}$, & ſignum ſuperius reperietur:

$$\text{Conſt.} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{A}{2.\,2^2} + \frac{B}{4.\,2^4} - \frac{C}{6.\,2^6} + \&c.\right)$$

$$\text{ſeu Conſt.} = \frac{3}{4} - \frac{A}{4.\,2^2} + \frac{B}{8.\,2^4} - \frac{C}{12.\,2^6} + \frac{D}{16.\,2^8} - \&c.$$

ſin autem ponatur $x = 4$, erit:

$$\text{Conſt.} = \frac{17}{24} - \frac{A}{4.\,4^2} + \frac{B}{8.\,4^4} - \frac{C}{12.\,4^6} + \frac{D}{16.\,4^8} - \&c.$$

Utcumque autem conſtans definiatur, idem prodibit valor, qui ſimul ſummam ſeriei in infinitum continuatae, quae eſt $= l\,2$, indicabit.

187. Ceterum ex his novis numeris A, B, C, D, E, &c. ſummae ſerierum poteſtatum reciprocarum parium, in quibus tantum numeri impares occurrunt, commode ſummari poterunt. Si enim ponatur:

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \&c. = s \quad \text{erit}$$

$$\frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{6^{2n}} + \&c. = \frac{s}{2^{2n}},$$

quae ab illa ſubtracta relinquet:

$$1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \&c. = \frac{(2^{2n}-1)s}{2^{2n}}.$$

Cum

Cum igitur valores ipſius s pro ſingulis numeris n iam ſupra exhibuerimus: (125), erit:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\&c.=\frac{A}{1.2}\cdot\frac{\pi^2}{4}$$

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\&c.=\frac{B}{1.2.3.4}\cdot\frac{\pi^4}{4}$$

$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\&c.=\frac{C}{1.2.3\ldots6}\cdot\frac{\pi^6}{4}$$

$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\&c.=\frac{D}{1.2.3\ldots8}\cdot\frac{\pi^8}{4}$$

$$1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\&c.=\frac{E}{1.2.3\ldots10}\cdot\frac{\pi^{10}}{4}$$

&c.

Sin autem omnes numeri ingrediantur, ſignaque alternentur quia erit:

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\&c.=\frac{(2^{2n}-1)s-s}{2^{2n}}$$

habebitur:

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\&c.=\frac{(A-2\mathfrak{A})}{1.2}\cdot\frac{\pi^2}{4}=\frac{(2-1)\mathfrak{A}}{1.2}\cdot\pi^2$$

$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\&c.=\frac{(B-2\mathfrak{B})}{1.2.3.4}\cdot\frac{\pi^4}{4}=\frac{(2^3-1)\mathfrak{B}}{1.2.3.4}\cdot\pi^4$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\&c.=\frac{(C-2\mathfrak{C})}{1.2\ldots6}\cdot\frac{\pi^6}{4}=\frac{(2^5-1)\mathfrak{C}}{1.2\ldots6}\cdot\pi^6$$

$$1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\&c.=\frac{(D-2\mathfrak{D})}{1.2\ldots8}\cdot\frac{\pi^8}{4}=\frac{(2^7-1)\mathfrak{D}}{1.2\ldots8}\cdot\pi^8$$

$$1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\&c.=\frac{(E-2\mathfrak{E})}{1.2\ldots10}\cdot\frac{\pi^{10}}{4}=\frac{(2^9-1)\mathfrak{E}}{1.2\ldots10}\cdot\pi^{10}$$

&c.

188. Quemadmodum hactenus ſeriem ſumus contemplati, cuius termini erant producta ex terminis progreſſionis

 geome-

geometricae p, p^2, p^3, &c. & ex terminis ſeriei cuiuſcunque a, b, c, &c. ita poterimus ſimili ratione proſequi ſeriem, cuius termini ſint producta ex terminis duarum quarumcunque ſerierum, quarum altera tanquam cognita aſſumatur.

$$\begin{array}{ll} \text{Sit ſeries cognita:} & \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \ \ldots \ + \overset{x}{Z} \\ \text{altera vero incognita} & a + b + c + \ \ldots \ + z \end{array}$$

atque quaeratur ſumma huius ſeriei:

$Aa + Bb + Cc + \ldots + Zz$

quae ponatur $= Zs$. Sit in ſerie cognita terminus penultimus $= Y$, atque poſito $x - 1$ loco x expreſſio ſummae $S. Zs$ abibit in

$$Y\left(s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \&c.\right)$$

Quae cum exprimat ſummam ſeriei Zs termino ultimo Zz multatae erit:

$$Zs - Zz = Ys - \frac{Yds}{dx} + \frac{Ydds}{2dx^2} - \frac{Yd^3s}{6dx^3} + \&c.$$

quae aequatio continet relationem, qua ſumma Zs pendet ab Y, Z, & z.

189. Ad hanc aequationem reſolvendam negligantur primum termini differentiales, eritque $s = \frac{Zz}{Z - Y}$, ponatur iſte valor $\frac{Zz}{Z - Y} = P^1$, ſitque revera $s = P^1 + p$, quo valore in aequatione ſubſtituto fiet:

$$\begin{aligned}(Z - Y)p = & -\frac{YdP^1}{dx} + \frac{YddP^1}{2dx^2} - \&c. \\ & -\frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \&c.\end{aligned}$$

addatur utrinque YP^1, & cum $P^1 - \frac{dP^1}{dx} + \frac{ddP^1}{2dx^2} - \&c.$

ſit valor ipſius P^1, qui prodit ſi loco x ponatur $x-1$, ſit iſte valor $= P$, eritque

$$(Z-Y)p + YP^1 = YP - \frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \&c.$$

unde neglectis differentialibus erit: $p = \frac{Y(P-P^1)}{Z-Y}$.

Ponatur $\frac{Y(P-P^1)}{Z-Y} = Q^1$, ſitque $p = Q^1 + q$; fiet

$$(Z-Y)q = -\frac{Y(dQ^1+dq)}{dx} + \frac{Y(ddQ^1+ddq)}{2dx^2} - \&c.$$

poſitoque Q pro valore ipſius Q^1, quem induit ſi loco x ſcribatur $x-1$, erit:

$$(Z-Y)q + YQ^1 = YQ - \frac{Ydq}{dx} + \frac{Yddq}{2dx^2} - \&c.$$

unde neglectis differentialibus fit $q = \frac{Y(Q-Q^1)}{Z-Y}$.

Ponatur $\frac{Y(Q-Q^1)}{Z-Y} = R^1$, ſitque revera $q = R^1 + r$;

ac ſimili modo reperitur $r = \frac{Y(R-R^1)}{Z-Y}$, ſicque procedendo erit ſumma quaeſita: $Zs = Z(P^1 + Q^1 + R^1 + \&c.)$.

190. Propoſita ergo ſerie quacunque:

$$Aa + Bb + Cc + \ldots\ldots + Yy + Zz$$

eius ſumma ſequenti modo definietur:

Ponatur poſito $x-1$ loco x

$\frac{Zz}{Z-Y} = P^1$; abeatque P^1 in P

$\frac{Y(P-P^1)}{Z-Y} = Q^1$; abeatque Q^1 in Q

$\frac{Y(Q-Q^1)}{Z-Y} = R^1$; abeatque R^1 in R

Y

$$\frac{Y(R-R^{I})}{Z-Y} = S^{I}; \text{ abeatque } S^{I} \text{ in } S$$

&c.

His valoribus inventis erit ſumma ſeriei =

$$ZP^{I} + ZQ^{I} + ZR^{I} + ZS^{I} + \&c.$$

+ Conſtante, quae reddat ſummam $= 0$, ſi ponatur $x = 0$, ſeu quod eodem redit, quae efficiat, ut cuipiam caſui ſatisfiat.

191. Formula haec, quia nullis differentialibus eſt implicata, in plurimis caſibus facillime adhibetur, atque etiam veram ſummam ſaepenumero exhibet. Sic ſi proponatur haec ſeries:

$$p + 4p^2 + 9p^3 + 16p^4 + \;.\;.\;.\;.\; + x^2 p^x$$

fiat $Z = p^x$ & $z = x^2$, erit $Y = p^{x-1}$, atque

$$\frac{Z}{Z-Y} = \frac{p}{p-1}, \quad \& \quad \frac{Y}{Z-Y} = \frac{1}{p-1}, \quad \text{Hinc fiet}$$

$$P^{I} = \frac{px^2}{p-1}; \qquad P = \frac{pxx - 2px + p}{p-1}$$

$$Q^{I} = -\frac{2px+p}{(p-1)^2}; \qquad Q = -\frac{2px + 3p}{(p-1)^2}$$

$$R^{I} = \frac{2p}{(p-1)^3}; \qquad R = \frac{2p}{(p-1)^3}$$

$$S^{I} = 0; \qquad \text{\& reliqui evaneſcunt omnes:}$$

unde erit ſumma =

$$p^x\left(\frac{px^2}{p-1} - \frac{2px+p}{(p-1)^2} + \frac{2p}{(p-1)^3}\right) - \frac{p}{(p-1)^2} - \frac{2p}{(p-1)^3}$$

$$= p^{x+1}\left(\frac{x^2}{p-1} - \frac{2x}{(p-1)^2} + \frac{p+1}{(p-1)^3}\right) - \frac{p+1}{(p-1)^3},$$

quemadmodum iam ſupra invenimus.

192. Simili modo, quo ad hanc ſummae expreſſionem pervenimus, aliam invenire poterimus expreſſionem, ſi ſeries propoſita non ex duabus aliis ſit compoſita: quae illis potiſſimum

ſimum caſibus in uſum vocari poterit, cum in praecedente expreſſione ad denominatores evaneſcentes pervenitur. Sit igitur propoſita haec ſeries:

$$s = \overset{1}{a} + \overset{2}{b} + \overset{3}{c} + \overset{4}{d} + \quad . \quad . \quad . \quad . \quad + \overset{x}{z}$$

quoniam poſito $x - 1$ loco x, ſumma ultimo termino truncatur, erit:

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \&c.$$

ſeu
$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \&c.$$

Quia hic ipſa ſumma s non occurrit, negligantur differentialia altiora, fietque $s = \int z dx$, ponatur $\int z dx = P'$, cuius valor abeat in P ſi pro x ſcribatur $x - 1$: ſitque revera $s = P' + p$, erit:

$$z = \frac{dP'}{dx} - \frac{ddP'}{2dx^2} + \&c. + \frac{dp}{dx} - \frac{ddp}{2dx^2} + \&c.$$

quia eſt $P = P' - \frac{dP'}{dx} + \frac{ddP'}{2dx^2} - \&c.$

erit $z - P' + P = \frac{dp}{dx} - \frac{ddp}{2dx^2} + \&c.$ unde fit $p = \int (z - P' + P) dx$. Si porro ponatur $\int (z - P' + P) dx = Q'$, hicque valor abeat in Q poſito $x - 1$ loco x, fit $\int (z - P' + P - Q' + Q) dx = R' = Q' - \int (Q' - Q) dx$ porro $R' - \int (R' - R) dx = S'$; &c. erit ſumma quaeſita: $s = P' + Q' + R' + S' + \&c. + \text{Con.}$ qua uni caſui ſatisfiat.

193. Mutatis aliquantum litteris iſta ſummatio huc redit. Propoſita ſerie ſummanda:

$$s = \overset{1}{a} + \overset{2}{b} + \overset{3}{c} + \overset{4}{d} + \quad . \quad . \quad . \quad . \quad + \overset{x}{z}$$

ponatur	*poſito* $x - 1$ *loco* x
$\int z dx = P$	abeatque P in p
$P - \int (P - p) dx = Q$	abeatque Q in q
$Q - \int (Q - q) dx = R$	abeatque R in r

quibus valoribus inventis erit ſumma quaeſita:

$$s = P + Q + R + S + \&c.$$

haecque expreſſio expedite oſtendit ſummam, ſi formulae iſtae integrales exhiberi queant. Sit, ut uſum eius exemplo illuſtremus, $z = xx + x$, eritque

$$P = \tfrac{1}{3}x^3 + \tfrac{1}{2}xx;\quad p = \tfrac{1}{3}x^3 - \tfrac{1}{2}xx + \tfrac{1}{6}$$

$$P - p = xx - \tfrac{1}{6}\ \&\ \int(P - p)\,dx = \tfrac{1}{3}x^3 - \tfrac{1}{6}x$$

$$Q = \tfrac{1}{2}xx + \tfrac{1}{6}x;\quad q = \tfrac{1}{2}xx - \tfrac{1}{6}x + \tfrac{1}{3}$$

$$Q - q = x - \tfrac{1}{3}\ \&\ \int(Q - q)\,dx = \tfrac{1}{2}xx - \tfrac{1}{3}x$$

$$R = \tfrac{1}{2}x\quad;\quad r = \tfrac{1}{2}x - \tfrac{1}{2}$$

$$R - r = \tfrac{1}{2}\quad;\quad \int(R - r)\,dx = \tfrac{1}{2}x$$

$S = 0$, reliquique valores evaneſcunt. Quare ſumma quaeſita erit:

$$\tfrac{1}{3}x^3 + \tfrac{1}{2}xx + \tfrac{1}{2}xx + \tfrac{1}{6}x + \tfrac{1}{2}x = \tfrac{1}{3}x^3 + xx + \tfrac{2}{3}x = \tfrac{1}{3}x(x+1)(x+2).$$

Hocque ergo modo omnium ſerierum, quarum termini generales ſunt functiones rationales integrae ipſius x, ſummae ope integrationum continuarum inveniri poſſunt. Ex quibus facile perſpicitur, quam amplum occupet campum doctrina de ſummatione ſerierum, neque omnibus methodis, quae tum habentur tum adhuc excogitari poſſunt, capiendis plura volumina ſufficere.

194. Hactenus ſummas ſerierum inveſtigavimus a termino primo uſque ad eum cuius index eſt x, quibus cognitis ponendo $x = \infty$ ipſius ſeriei in infinitum continuatae ſumma innoteſcet. Saepenumero autem hoc expeditius praeſtatur, ſi non ſumma terminorum a primo uſque ad eum cuius index eſt x, ſed ſumma omnium terminorum ab iſto, cuius index eſt x, in infinitum uſque quaeratur, hocque caſu imprimis expreſſiones ultimae fiunt tractabiliores. Sit igitur propoſita ſeries cuius terminus generalis ſeu indici x reſpondens ſit $= z$, ſequens indici $x + 1$ reſpondens ſit $= z^{\mathrm{I}}$, huncque ultra ſequentes ſint z^{II}, z^{III}, &c. quaeraturque ſumma huius ſeriei infinitae:

x,

$x,\ x+1,\ x+2,\ x+3,$ &c.

$s = z + z^{I} + z^{II} + z^{III} +$ &c. in infinitum.

Haec igitur ſumma s erit functio ipſius x, in qua ſi ponatur $x+1$ loco x, orietur ſumma prior termino z truncata. Cum ergo hac mutatione s abeat in

$$s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \text{\&c.} \quad \text{erit:}$$

$$s - z = s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{\&c.}$$

$$\text{ſeu } 0 = z + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{\&c.}$$

195. Si nunc ut ante ratiocinium inſtituamus, fiet neglectis differentialibus ſuperioribus, $s = C - \int z dx$. Ponatur ergo $\int z dx = P$, ſitque revera $s = C - P + p$, erit

$$0 = z - \frac{dP}{dx} - \frac{ddP}{2dx^2} - \frac{d^3P}{6dx^3} - \text{\&c.}$$
$$+ \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{\&c.}$$

Abeat P in P^{I}, ſi loco x ponatur $x+1$, eritque

$$0 = z + P - P^{I} + \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{\&c.}$$

Hinc neglectis differentialibus altioribus fiet:
$p = \int(P^{I} - P)dx - P$. Statuatur $\int(P^{I} - P)dx - P = -Q$, ſitque $p = -Q + q$, erit:

$$0 = z + P - P^{I} - \frac{dQ}{dx} - \frac{ddQ}{2dx^2} - \text{\&c.} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{\&c.}$$

Abeat Q in Q^{I}, ſi loco x ponatur $x+1$, eritque:

$$0 = z + P - P^{I} + Q - Q^{I} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{\&c.}$$

unde ſequitur $q = \int(Q^{I} - Q)\,dx - Q$. Quamobrem ſi comma cuique quantitati infixum denotet eius valorem, quem induit poſito $x+1$ loco x, ponaturque

$$\int z\,dx = P$$
$$P - \int (P^{I} - P)\,dx = Q$$
$$Q - \int (Q^{I} - Q)\,dx = R$$
$$R - \int (R^{I} - R)\,dx = S \quad \&c.$$

erit ſeriei propoſitae $z + z^{I} + z^{II} + z^{III} + z^{IV} + \&c.$ ſumma $= C - P - Q - R - S - \&c.$ ubi conſtans C ita debet definiri, ut poſito $x = \infty$ tota ſumma evaneſcat. Quia autem applicatio huius expreſſionis integrationes requirit, hoc loco eius uſum declarare non licet.

196. Ut autem formulas integrales evitemus, ſtatuamus ſummam ſeriei $= y\,s$, exiſtente y functione ipſius x quacunque cognita, cuius valores y^{I}, y^{II}, &c. qui prodeunt ponendo $x + 1$, $x + 2$, &c. loco x, erunt noti. Si iam ponatur $x + 1$ loco x prodibit ſuperior ſeries termino primo mulctata, cuius ſumma propterea erit

$$y'\left(s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3 s}{6dx^3} + \&c.\right) = ys - z$$

ſeu
$$z + \frac{y'ds}{dx} + \frac{y'dds}{2dx^2} + \frac{y'd^3 s}{6dx^3} + \&c. = (y - y')\,s$$

unde neglectis differentialibus oritur $s = \frac{z}{y - y'}$. Statuatur $\frac{z}{y' - y} = P$, ſitque revera $s = -P + p$, erit

$$\left.\begin{array}{l} -\frac{y'dP}{dx} - \frac{y'ddP}{2dx^2} - \frac{y'd^3 P}{6dx^3} - \&c. \\ +\frac{y'dp}{dx} + \frac{y'ddp}{2dx^2} + \frac{y'd^3 p}{6dx^3} + \&c. \end{array}\right\} = (y - y')\,p$$

ideoque $\frac{y'dp}{dx} + \frac{y'ddp}{2dx^2} + \frac{y'd^3 p}{6dx^3} + \&c. = y'(P' - P) - (y' - y)p.$

Statuatur $Q = \frac{y'(P' - P)}{y' - y}$, ſitque $p = Q + q$; erit:

$$y'(Q' - Q) + y'\left(\frac{dq}{dx} + \frac{ddq}{2dx^2} + \&c.\right) = -(y' - y)\,q.$$

Statuatur $R = \frac{y'(Q' - Q)}{y' - y}$, ſitque $q = -R + r$.

Hocque modo ſi ulterius progrediamur. Seriei propoſitae:

$$z + z^{I} + z^{II} + z^{III} + z^{IV} + \&c.$$

ſumma ſequenti modo invenietur.

Sumta pro lubitu functione ipſius x, quae ſit $= y$, ſtatuatur:

$$P = \frac{z}{y' - y} = \frac{z}{\Delta y}$$

$$Q = \frac{y'(P' - P)}{y' - y} = \frac{y\Delta P}{\Delta y} + \Delta P$$

$$R = \frac{y'(Q' - Q)}{y' - y} = \frac{y\Delta Q}{\Delta y} + \Delta Q$$

$$S = \frac{y'(R' - R)}{y' - y} = \frac{y\Delta R}{\Delta y} + \Delta R \quad \&c.$$

Hincque erit ſumma quaeſita:

$$= C - Py + Qy - Ry + Sy - \&c.$$

Sumta pro C eiuſmodi conſtante, ut poſito $x = \infty$ ſumma evaneſcat.

197. Sumatur $y = a^x$, ob $y' = a^{x+1}$, erit: $y' - y = a^x(a - 1)$, unde fiet:

$$P = \frac{z}{a^x(n-1)} \; ; \quad P' = \frac{z'}{a^{x+1}(a-1)}$$

$$Q = \frac{a(P' - P)}{a - 1} = \frac{z' - az}{a^x(a-1)^2} \; ; \quad Q' = \frac{z'' - az'}{a^{x+1}(a-1)^2}$$

$$R = \frac{a(Q' - Q)}{a - 1} = \frac{z'' - 2az' + aaz}{a^x(a-1)^3}$$

$$S = \frac{a(R' - R)}{a - 1} = \frac{z''' - 3az'' + 3a^2z' - a^3z}{a^x(a-1)^4} \quad \&c.$$

Quocirca ſumma ſeriei propoſitae erit:

$$C - \frac{z}{a-1} + \frac{z' - az}{(a-1)^2} - \frac{z'' + 2az' - a^2z}{(a-1)^3} + \frac{z''' - 3az'' + 3a^2z' - a^3z}{(a-1)^4} \&c.$$

Haec vero eadem ſummae expreſſio iam ſupra eſt inventa Capite primo. Hinc autem aliis pro y valoribus accipiendis infinitae aliae expreſſiones erui poterunt; unde ea, quae cuique caſui maxime ſit accommodata, eligi poteſt.

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CAPUT VIII.

DE USU CALCULI DIFFERENTIALIS IN FORMANDIS SERIEBUS.

198.

Unum adhuc calculi differentialis uſum in doctrina ſerierum commemorabimus, qui in ipſa formatione ſerierum conſiſtit, & ad quem iam ſupra provocavimus, cum quaeſtio eſſet de fractione, cuius denominator ſit poteſtas quaecunque functionis cuiuſpiam, in ſeriem evolvenda. Iſta methodus autem ſimilis eſt ei, qua iam aliquoties ſumus uſi, dum functio in ſeriem convertenda aequalis fingitur cuipiam ſeriei, in ſingulis terminis coefficientes indeterminatos habenti, qui deinceps aequalitate conſtituta determinentur. Haec autem determinatio ſaepenumero mirifice adiuvatur, ſi antequam ea ſuſcipiatur ad differentialia cum prima, tum nonnunquam quoque ad ſecunda aequatio perducatur. Quae methodus cum in calculo integrali ampliſſimi ſit uſus, eam hic diligentius exponemus.

199. Primum igitur breviter repetamus, quae ſupra de evolutione fractionum in ſeries ſine calculi differentialis ſubſidio attulimus. Sit fractio quaecunque propoſita:

$$\frac{A + Bx + Cx^2 + Dx^3 + \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.} = s$$

quam in ſeriem ſecundum poteſtates ipſius x procedentem converti oporteat. Fingatur pro s ſeries indeterminata:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \&c.$$

Cum igitur fractione per multiplicationem ſublata ſit:

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 \quad \&c.$$
$$= s(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \eta x^6 \quad \&c.)$$

ſi pro s ſeries ficta ſubſtituatur prodibit ſequens aequatio:

A

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c. =$$

$$\begin{array}{llllll l}
\mathfrak{A}\alpha + \mathfrak{B}\alpha x & + \mathfrak{C}\alpha x^2 & + \mathfrak{D}\alpha x^3 & + \mathfrak{E}\alpha x^4 & + \mathfrak{F}\alpha x^5 & + \&c. \\
+ \mathfrak{A}\beta & + \mathfrak{B}\beta & + \mathfrak{C}\beta & + \mathfrak{D}\beta & + \mathfrak{E}\beta & + \&c. \\
 & + \mathfrak{A}\gamma & + \mathfrak{B}\gamma & + \mathfrak{C}\gamma & + \mathfrak{D}\gamma & + \&c. \\
 & & + \mathfrak{A}\delta & + \mathfrak{B}\delta & + \mathfrak{C}\delta & + \&c. \\
 & & & + \mathfrak{A}\varepsilon & + \mathfrak{B}\varepsilon & + \&c. \\
 & & & & + \mathfrak{A}\zeta & + \&c.
\end{array}$$

Aequalitate ergo inter ſingulos terminos, qui eaſdem ipſius x poteſtates continent, conſtituta fiet:

$$\mathfrak{A}\alpha - A = 0$$
$$\mathfrak{B}\alpha + \mathfrak{A}\beta - B = 0$$
$$\mathfrak{C}\alpha + \mathfrak{B}\beta + \mathfrak{A}\gamma - C = 0$$
$$\mathfrak{D}\alpha + \mathfrak{C}\beta + \mathfrak{B}\gamma + \mathfrak{A}\delta - D = 0$$
$$\mathfrak{E}\alpha + \mathfrak{D}\beta + \mathfrak{C}\gamma + \mathfrak{B}\delta + \mathfrak{A}\varepsilon - E = 0 \quad \&c.$$

ex quibus aequationibus coefficientes ficti $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. determinantur, ſicque ſeries infinita invenitur:

$$\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$$

fractioni propoſitae s aequalis. Atque in hac forma ſi tam numerator quam denominator fractionis s finito terminorum numero conſtent, omnes ſeries recurrentes comprehenduntur, de quibus iam ſupra fuſius eſt tractatum.

200. Quodſi autem vel numerator vel denominator vel uterque ad dignitatem quamcunque fuerit elevatus, tum hoc modo ſeries difficulter obtinetur; propterea quod negotium, niſi functio elevata ſit binomium, perquam ſit laborioſum. Calculo autem differentiali iſte labor evitari poteſt. Adſit primum ſolus numerator, ſitque:

$$s = (A + Bx + Cxx)^n,$$

unde quaeratur ſeries ſecundum poteſtates ipſius x procedens huic trinomii dignitati aequalis; quam quidem finitam fore conſtat, ſi exponens n fuerit numerus integer affirmativus. Fingatur iterum pro s ſeries indefinita:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \&c.$$

cuius terminum primum $\mathfrak{A}$ conſtat eſſe $= A^n$: ſi enim ponatur

tur $x=0$, ex priori forma propoſita fit $s = A^n$, ex ſerie autem ficta $s = \mathfrak{A}$. Haec autem primi termini determinatio ex ipſa rei natura eſt petenda, ſi ad differentialia deſcendere velimus, quia hinc primus terminus non determinatur, uti mox patebit.

201. Cum ſit $s = (A + Bx + Cx^2)^n$, erit logarithmis ſumendis $ls = nl(A + Bx + Cx^2)$, hincque ſumtis differentialibus habebitur: $\frac{ds}{s} = \frac{nBdx + 2nCxdx}{A + Bx + Cx^2}$, ſeu

$$(A + Bx + Cx^2)\frac{ds}{dx} = ns(B + 2Cx).$$

Ex ſerie autem ficta eſt:

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \&c.$$

ſi igitur haec ſeries loco $\frac{ds}{dx}$, & pro s ipſa ſeries ficta ſubſtituatur, prodibit ſequens aequatio:

$$\begin{array}{llllll}
A\mathfrak{B} + & 2A\mathfrak{C}x + & 3A\mathfrak{D}x^2 & + 4A\mathfrak{E}x^3 & + 5A\mathfrak{F}x^4 & + \&c. \\
+ & B\mathfrak{B} + & 2B\mathfrak{C} & + 3B\mathfrak{D} & + 4B\mathfrak{E} & + \&c. \\
 & + & C\mathfrak{B} & + 2C\mathfrak{C} & + 3C\mathfrak{D} & + \&c. \\
\hline
nB\mathfrak{A} + & nB\mathfrak{B} + & nB\mathfrak{C} & + nB\mathfrak{D} & + nB\mathfrak{E} & + \&c. \\
+ 2nC\mathfrak{A} + & 2nC\mathfrak{B} & + 2nC\mathfrak{C} & + 2nC\mathfrak{D} & & + \&c.
\end{array}$$

Aequalitate ergo hic inter terminos eiuſdem ipſius x poteſtatis conſtituta erit:

$$\begin{aligned}
\mathfrak{B} &= \frac{nB\mathfrak{A}}{A} \\
\mathfrak{C} &= \frac{(n-1)B\mathfrak{B} + 2nC\mathfrak{A}}{2A} \\
\mathfrak{D} &= \frac{(n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B}}{3A} \\
\mathfrak{E} &= \frac{(n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C}}{4A} \\
\mathfrak{F} &= \frac{(n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D}}{5A} \qquad \&c.
\end{aligned}$$

Cum igitur ut ante vidimus ſit $\mathfrak{A} = A^n$, erit $\mathfrak{B} = nA^{n-1}B$, hincque reliqui coefficientes omnes ſucceſſive determinabuntur. Lex autem, quam ipſi ſequuntur facillime ex his formulis patet, quae vehementer obſcura manſiſſet, ſi trinomium actu elevare voluiſſemus.

202. Haec eadem methodus ſuccedit, ſi polynomium quodcunque ad quampiam dignitatem elevari debeat. Sit $s = (A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)^n$ fingaturque:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$$

erit $\mathfrak{A} = A^n$, qui valor colligitur, ſi ponatur $x = 0$. Sumtis iam ut ante logarithmis, eorumque differentialibus reperietur:

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx + 3nDx^2dx + 4nEx^3dx + \&c.}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.}$$

ſeu $(A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)\frac{ds}{dx} =$

$s(nB + 2nCx + 3nDx^2 + 4nEx^3 + \&c.)$. Cum igitur ſit:

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \&c.$$

Erit his ſeriebus pro s & $\frac{ds}{dx}$ ſubſtitutis:

$$\begin{array}{llllll}
A\mathfrak{B} & + 2A\mathfrak{C}x & + 3A\mathfrak{D}x^2 & + 4A\mathfrak{E}x^3 & + 5A\mathfrak{F}x^4 & + \&c. \\
 & + B\mathfrak{B} & + 2B\mathfrak{C} & + 3B\mathfrak{D} & + 4B\mathfrak{E} & + \&c. \\
 & & + C\mathfrak{B} & + 2C\mathfrak{C} & + 3C\mathfrak{D} & + \&c. \\
 & & & + D\mathfrak{B} & + 2D\mathfrak{C} & + \&c. \\
 & & & & + E\mathfrak{B} & + \&c. = \\
\hline
nB\mathfrak{A} & + nB\mathfrak{B} & + nB\mathfrak{C} & + nB\mathfrak{D} & + nB\mathfrak{E} & + \&c. \\
 & + 2nC\mathfrak{A} & + 2nC\mathfrak{B} & + 2nC\mathfrak{C} & + 2nC\mathfrak{D} & + \&c. \\
 & & + 3nD\mathfrak{A} & + 3nD\mathfrak{B} & + 3nD\mathfrak{C} & + \&c. \\
 & & & + 4nE\mathfrak{A} & + 4nE\mathfrak{B} & + \&c. \\
 & & & & + 5nF\mathfrak{A} & + \&c.
\end{array}$$

Unde derivantur ſequentes determinationes:

$$\begin{array}{l}
A\mathfrak{B} = n\,B\mathfrak{A} \\
2A\mathfrak{C} = (n-1)B\mathfrak{B} + 2nC\mathfrak{A} \\
3A\mathfrak{D} = (n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B} + 3nD\mathfrak{A} \\
4A\mathfrak{E} = (n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C} + (3n-1)D\mathfrak{B} + 4nE\mathfrak{A}
\end{array}$$

$$5A\mathfrak{F} = (n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D} + (3n-2)D\mathfrak{C} + (4n-1)E\mathfrak{B} + 5nF\mathfrak{A}$$
&c.

unde quemadmodum coefficientes ficti $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. a ſe invicem pendeant, hincque determinentur, cum ſit $\mathfrak{A} = A^n$, luculentiſſime apparet.

203. Quoniam, ſi quantitas $A + Bx + Cx^2 + Dx^3 +$ &c. ex finito terminorum numero conſtat, numeruſque n fuerit integer affirmativus, quaecunque poteſtas finito etiam terminorum numero conſtare debet: manifeſtum eſt hoc caſu, formulas modo inventas tandem evaneſcere debere, atque cum omnes termini adeſſe debeant, ut primum unus evanuerit, ſimul omnes ſequentes evaneſcere debere. Ponamus formulam propoſitam $A + Bx + Cx^2$ eſſe trinomium, eiuſque cubum quaeri, ut ſit $n = 3$, erit

$$\begin{array}{rcll}
\mathfrak{A} &=& A^3 \quad \text{ideoque} ; & \mathfrak{A} = A^3 \\
A\mathfrak{B} &=& 3B\mathfrak{A} ; & \mathfrak{B} = 3A^2B \\
2A\mathfrak{C} &=& 2B\mathfrak{B} + 6C\mathfrak{A} ; & \mathfrak{C} = 3AB^2 + 3A^2C \\
3A\mathfrak{D} &=& 1B\mathfrak{C} + 5C\mathfrak{B} ; & \mathfrak{D} = B^3 + 6ABC \\
4A\mathfrak{E} &=& 0 + 4C\mathfrak{C} ; & \mathfrak{E} = 3B^2C + 3AC^2 \\
5A\mathfrak{F} &=& -B\mathfrak{E} + 3C\mathfrak{D} ; & \mathfrak{F} = 3BC^2 \\
6A\mathfrak{G} &=& -2B\mathfrak{F} + 2C\mathfrak{E} ; & \mathfrak{G} = C^3 \\
7A\mathfrak{H} &=& -3B\mathfrak{G} + 1C\mathfrak{F} ; & \mathfrak{H} = 0 \\
8A\mathfrak{I} &=& -4B\mathfrak{H} + 0 ; & \mathfrak{I} = 0.
\end{array}$$

Quoniam igitur iam bini ſunt $= 0$, ſequentiumque quilibet a duobus praecedentibus pendet, patet omnes ſequentes pariter evaneſcere debere. Hancque ob cauſam lex, qua hi coefficientes a ſe invicem pendere ſunt inventi, eo magis eſt notatu digna.

204. Si n fuerit numerus negativus, ita ut s aequale fiat fractioni, ſeries in infinitum excurret. Sit igitur

$$s = \frac{1}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingatur pro eius valore haec ſeries:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \&c.$$

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Atque ſi in ſuperioribus formulis pro litteris A, B, C, D, &c. ponantur α, β, γ, δ, &c. ſimulque fiat n negativum, ſequentes determinationes coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. prodibunt:

$$\mathfrak{A} = \alpha^{-n} = \frac{1}{\alpha^n}$$

$$\alpha\mathfrak{B} + n\beta\mathfrak{A} = 0$$

$$2\alpha\mathfrak{C} + (n+1)\beta\mathfrak{B} + 2n\gamma\mathfrak{A} = 0$$

$$3\alpha\mathfrak{D} + (n+2)\beta\mathfrak{C} + (2n+1)\gamma\mathfrak{B} + 3n\delta\mathfrak{A} = 0$$

$$4\alpha\mathfrak{E} + (n+3)\beta\mathfrak{D} + (2n+2)\gamma\mathfrak{C} + (3n+1)\delta\mathfrak{B} + 4n\varepsilon\mathfrak{A} = 0$$

$$5\alpha\mathfrak{F} + (n+4)\beta\mathfrak{E} + (2n+3)\gamma\mathfrak{D} + (3n+2)\delta\mathfrak{C} + (4n+1)\varepsilon\mathfrak{B} + 5n\zeta\mathfrak{A} = 0$$

&c.

Quae formulae eandem continent legem horum coefficientium numerorum, quam iam ſupra obſervavimus in Introductione; cuiuſque adeo veritatem nunc demum rigide demonſtrare licuit.

205. Haec ita ſe habent, ſi numerator fractionis fuerit unitas, vel etiam quaepiam ipſius x poteſtas, puta x^m; poſteriori enim caſu tantum oportebit ſeriem priori inventam $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 +$ &c. multiplicare per x^m. At ſi numerator conſtet ex duobus pluribuſve terminis, tum ſupra quidem legem progreſſionis non obſervavimus, quamobrem eam hic per differentiationem inveſtigemus. Sit igitur:

$$s = \frac{A + Bx + Cx^2 + Dx^3 + \&c.}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingaturque pro valore huius fractionis ſequens ſeries:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \&c.$$

cuius primus terminus $\mathfrak{A}$ ut definiatur, ponatur $x = 0$; eritque ex priori expreſſione $s = \frac{A}{\alpha^n}$, ex ficta vero $s = \mathfrak{A}$, unde neceſſe eſt, ut ſit $\mathfrak{A} = \frac{A}{\alpha^n}$. Quo termino determinato reliqui per differentiationem innoteſcent.

206. Sumtis logarithmis erit;

$$ls = l(A + Bx + Cx^2 + Dx^3 + \&c.)$$
$$- nl(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)$$

hincque differentiando orietur:

$$\frac{ds}{s} = \frac{Bdx + 2Cxdx + 3Dx^2dx + \&c.}{A + Bx + Cx^2 + Dx^3 + \&c.} - \frac{n\beta dx - 2n\gamma xdx - 3n\delta x^2dx - \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.}$$

ſublatiſque per multiplicationem fractionibus erit:

$$\left(\begin{array}{llll} A\alpha + A\beta x + A\gamma x^2 + A\delta x^3 + \&c. \\ \quad + B\alpha \; + B\beta \;\; + B\gamma \;\;\; + \&c. \\ \qquad\quad + C\alpha \;\; + C\beta \;\;\; + \&c. \\ \qquad\qquad\quad + D\alpha \;\;\; + \&c. \end{array}\right) \frac{ds}{dx} =$$

$$\left(\begin{array}{l} B\alpha + B\beta x + B\gamma x^2 + B\delta x^3 + B\varepsilon x^4 + \&c. \\ \quad + 2C\alpha + 2C\beta \;\; + 2C\gamma \;\; + 2C\delta \;\; + \&c. \\ \qquad\quad + 3D\alpha \;\; + 3D\beta \;\; + 3D\gamma \;\; + \&c. \\ \qquad\qquad\quad + 4E\alpha \;\; + 4E\beta \;\; + \&c. \\ \qquad\qquad\qquad\quad + 5F\alpha \;\; + \&c. \end{array}\right) s$$

$$-\left(\begin{array}{l} A\beta + 2A\gamma x + 3A\delta x^2 + 4A\varepsilon x^3 + 5A\zeta x^4 + \&c. \\ \quad + B\beta \;\; + 2B\gamma \;\; + 3B\delta \;\; + 4B\varepsilon \;\; + \&c. \\ \qquad\quad + C\beta \;\; + 2C\gamma \;\; + 3C\delta \;\; + \&c. \\ \qquad\qquad\quad + D\beta \;\; + 2D\gamma \;\; + \&c. \\ \qquad\qquad\qquad\quad + E\beta \;\; + \&c. \end{array}\right) ns.$$

Cum nunc ſit $\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + \&c.$

erit factis ſubſtitutionibus:

$$A\alpha\mathfrak{B} + nA\beta\mathfrak{A} - B\alpha\mathfrak{A} = 0$$

$$2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2nA\gamma\mathfrak{A} + (n-1)B\beta\mathfrak{A} - 2C\alpha\mathfrak{A} = 0$$

$$\left.\begin{array}{r} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3nA\delta\mathfrak{A} \\ B\alpha\mathfrak{C} + nB\beta\mathfrak{B} + (2n-1)B\gamma\mathfrak{A} \\ - C\alpha\mathfrak{B} + (n-2)C\beta\mathfrak{A} \\ - 3D\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{r} 4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + (2n+2)A\gamma\mathfrak{C} + (3n+1)A\delta\mathfrak{B} + 4nA\varepsilon\mathfrak{A} \\ 2B\alpha\mathfrak{D} + (n+1)B\beta\mathfrak{C} + 2nB\gamma\mathfrak{B} + (3n-1)B\delta\mathfrak{A} \\ 0\,C\alpha\mathfrak{C} + (n-1)C\beta\mathfrak{B} + (2n-2)C\gamma\mathfrak{A} \\ - 2D\alpha\mathfrak{B} + (n-3)D\beta\mathfrak{A} \\ - 4E\alpha\mathfrak{A} \end{array}\right\} = 0$$

Hinc lex; qua iſtae formulae progrediuntur, facile perſpicitur: prima enim cuiuſque aequationis linea eandem ſequitur legem, quam §. 204. habuimus. Tum vero coefficientes ſecundarum linearum oriuntur, ſi a coefficientibus ſuperioribus ſubtrahatur $n+1$, ſimilique modo ex linea ſecunda formatur linea tertia & ſequentes, a coefficientibus ſuperioribus continuo ſubtrahendo $n+1$; ipſae autem litterae quemvis terminum componentes per ſolam inſpectionem facillime formantur.

207. Sin autem quoque numerator fractionis fuerit quaepiam poteſtas: ſcilicet

$$s = \frac{(A + Bx + Cx^2 + Dx^3 + \&c.)^m}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingaturque $s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$

erit $\mathfrak{A} = \frac{A^m}{\alpha^n}$; reliqui vero coefficientes ex ſequentibus formulis determinabuntur:

$$\left.\begin{array}{l} A\alpha\mathfrak{B} + nA\beta\mathfrak{A} \\ \quad - mB\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2n\,A\gamma\mathfrak{A} \\ \quad - (m-1)B\alpha\mathfrak{B} + (n-m)B\beta\mathfrak{A} \\ \qquad - 2m\,C\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3n\,A\delta\mathfrak{A} \\ \quad - (m-2)B\alpha\mathfrak{C} + (n-m+1)B\beta\mathfrak{B} + (2n-m)B\gamma\mathfrak{A} \\ \qquad - (2m-1)C\alpha\mathfrak{B} + (n-2m)C\beta\mathfrak{A} \\ \qquad\quad - 3m\,D\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + (2n+2)A\gamma\mathfrak{C} + (3n+1)A\delta\mathfrak{B} + 4n\,A\varepsilon\mathfrak{A} \\ \quad - (m-3)B\alpha\mathfrak{D} + (n-m+2)B\beta\mathfrak{C} + (2n-m+1)B\gamma\mathfrak{B} + (3n-m)B\delta\mathfrak{A} \\ \qquad - (2m-2)C\alpha\mathfrak{C} + (n-2m+1)C\beta\mathfrak{B} + (2n-2m)C\gamma\mathfrak{A} \\ \qquad\quad - (3m-1)D\alpha\mathfrak{B} + (n-3m)D\beta\mathfrak{A} \\ \qquad\qquad - 4m\,E\alpha\mathfrak{A} \end{array}\right\} = 0$$

Lex, qua iſtae formulae ulterius continuantur, ex inſpectione facilius apparet, quam verbis deſcribi queat. Deſcendendo autem coefficientes diminuuntur differentia $n+m$; horizontaliter autem progrediendo augentur continuo differentia $n-1$,

208. Hoc igitur modo doctrina de ſeriebus recurrentibus amplificatur, dum iſtum defectum ſupplevimus, atque legem coefficientium definivimus, ſi non ſolum denominator fractionis fuerit poteſtas quaecunque, ſed etiam numerator ex quotlibet terminis conſtet, ad quam legem detegendam ſola inductio non ſufficiebat. Praeter plurimos autem uſus ſerierum recurrentium, quos iam expoſuimus, maximam quoque afferunt utilitatem ad ſummas quarumvis ſerierum proxime inveniendas: cuius ſpecimen iam in Capite primo huius ſectionis exhibuimus, dum ſeriem ſubſtitutione $x = \frac{y}{1 + ny}$ in aliam tranſmutavimus, quae ſaepenumero terminorum numero finito conſtet. Eaque methodus ulterius extendi potuiſſet, ſi pro x aliae functiones ſubſtitutae fuiſſent. Quoniam vero tum lex progreſſionis ſerierum, quae loco poteſtatum ipſius x poni deberent, non ſatis luculenter conſtabat, in hunc locum iſtam amplificationem reſervare viſum eſt; cum memorata lex iam penitus eſſet detecta. Interim tamen re diligentius perpenſa comperimus idem negotium ſine hac progreſſionis lege expediri poſſe, in ſubſidium tantum vocando methodum, qua hic ad hanc ipſam legem inveſtigandam ſumus uſi.

209. Sit igitur propoſita ſeries quaecunque

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

quam in aliam transformari oporteat, cuius termini ſinguli ſint fractiones, quarum denominatores ſecundum poteſtates formulae huiuſmodi $\alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.$ procedant. Quo igitur a ſimplicioribus incipiamus, ponamus eſſe:

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{B}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^3} + \frac{\mathfrak{D}x^3}{(\alpha + \beta x)^4} + \&c.$$

aequalitate illius ſeriei cum hac expreſſione conſtituta, multiplicetur ubique per $\alpha + \beta x$, fietque:

$$\begin{array}{llll} A\alpha + B\alpha x & + C\alpha x^2 & + D\alpha x^3 & + \&c. \\ \quad\; + A\beta + B\beta & + C\beta & + \&c. & \end{array} = \mathfrak{A} + \frac{\mathfrak{B}x}{\alpha + \beta x} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^2} + \&c.$$

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ſtatuatur $\mathfrak{A} = A\alpha$; fiatque:

$$A\beta + B\alpha = A^{I}$$
$$B\beta + C\alpha = B^{I}$$
$$C\beta + D\alpha = C^{I}$$
$$D\beta + E\alpha = D^{I} \quad \&c.$$

erit diviſione per x inſtituta:

$$A^{I} + B^{I}x + C^{I}x^2 + D^{I}x^3 + \&c. = \frac{\mathfrak{B}}{\alpha+\beta x} + \frac{\mathfrak{C}x}{(\alpha+\beta x)^2} + \frac{\mathfrak{D}x^2}{(\alpha+\beta x)^3} + \&c.$$

Multiplicetur denuo per $\alpha + \beta x$, poſitoque

$$A^{I}\beta + B^{I}\alpha = A^{II}$$
$$B^{I}\beta + C^{I}\alpha = B^{II}$$
$$C^{I}\beta + D^{I}\alpha = C^{II} \quad \&c. \quad \text{fiet}$$

$$A^{I}\alpha + A^{II}x + B^{II}x^2 + C^{II}x^3 + \&c. = \mathfrak{B} + \frac{\mathfrak{C}x}{\alpha+\beta x} + \frac{\mathfrak{D}x^2}{(\alpha+\beta x)^2} + \&c.$$

Sit igitur $\mathfrak{B} = A^{I}\alpha$; atque operationem ut ante inſtituendo, ſi fiat:

$A^{II}\beta + B^{II}\alpha = A^{III}$	$A^{III}\beta + B^{III}\alpha = A^{IV}$
$B^{II}\beta + C^{II}\alpha = B^{III}$	$B^{III}\beta + C^{III}\alpha = B^{IV}$
$C^{II}\beta + D^{II}\alpha = C^{III}$	$C^{III}\beta + D^{III}\alpha = C^{IV}$
&c.	&c.

erit $\mathfrak{C} = A^{II}\alpha$; $\mathfrak{D} = A^{III}\alpha$; $\mathfrak{E} = A^{IV}\alpha$; &c. unde ſumma ſeriei propoſitae hoc modo exprimetur, ut ſit:

$$s = \frac{A\alpha}{\alpha+\beta x} + \frac{A^{I}\alpha x}{(\alpha+\beta x)^2} + \frac{A^{II}\alpha x^2}{(\alpha+\beta x)^3} + \frac{A^{III}\alpha x^3}{(\alpha+\beta x)^4} + \&c.$$

Quae eadem ſeries orta fuiſſet ex ſubſtitutione $\frac{x}{\alpha+\beta x} = y$ ſeu $x = \frac{\alpha y}{1-\beta y}$.

210. Haec transformatio optimo cum ſucceſſu adhibetur, ſi ſeries propoſita $A + Bx + Cx^2 + \&c.$ ita fuerit comparata, ut tandem confundatur cum ſerie recurrente ſeu potius geometrica ex fractione $\frac{P}{\alpha+\beta x}$ orta. Tum enim valores A^{I},

A^{I}, B^{I}, C^{I}, D^{I}, &c. tandem evaneſcent; hincque multo magis litterae A^{II}, A^{III}, A^{IV}, &c. conſtituent ſeriem maxime convergentem. Poterimus autem ſimili modo denominatores trinomiales & polynomiales quoſcunque adhibere, qui uſum habebunt eximium, ſi ſeries propoſita tandem cum recurrente confundatur. Propoſita ergo ſerie:

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

ſtatuatur

$$s = \frac{\mathfrak{A} + \mathfrak{B}x}{\alpha + \beta x + \gamma x^2} + \frac{\mathfrak{A}^{I}x^2 + \mathfrak{B}^{I}x^3}{(\alpha + \beta x + \gamma x^2)^2} + \frac{\mathfrak{A}^{II}x^4 + \mathfrak{B}^{II}x^5}{(\alpha + \beta x + \gamma x^2)^3} + \frac{\mathfrak{A}^{III}x^6 + \mathfrak{B}^{III}x^7}{(\alpha + \beta x + \gamma x^2)^4} + \&c.$$

Multiplicetur ubique per $\alpha + \beta x + \gamma x^2$, poſitoque

$$\begin{aligned} A\gamma + B\beta + C\alpha &= A^{I} \\ B\gamma + C\beta + D\alpha &= B^{I} \\ C\gamma + D\beta + E\alpha &= C^{I} \\ &\&c. \end{aligned} \quad \& \quad \begin{aligned} \mathfrak{A} &= A\alpha \\ \mathfrak{B} &= A\beta + B\alpha \end{aligned}$$

orietur aequatio priori ſimilis, diviſione per xx inſtituta.

$$A^{I} + B^{I}x + C^{I}x^2 + D^{I}x^3 + E^{I}x^4 + \&c. =$$

$$\frac{\mathfrak{A}^{I} + \mathfrak{B}^{I}x}{\alpha + \beta x + \gamma xx} + \frac{\mathfrak{A}^{II}x^2 + \mathfrak{B}^{II}x^3}{(\alpha + \beta x + \gamma xx)^2} + \frac{\mathfrak{A}^{III}x^4 + \mathfrak{B}^{III}x^5}{(\alpha + \beta x + \gamma xx)^3} + \&c.$$

Si igitur ut ante operatio inſtituatur faciendo

$$\begin{aligned} A^{I}\gamma + B^{I}\beta + C^{I}\alpha &= A^{II} \\ B^{I}\gamma + C^{I}\beta + D^{I}\alpha &= B^{II} \\ C^{I}\gamma + D^{I}\beta + E^{I}\alpha &= C^{II} \\ &\&c. \end{aligned} \qquad \begin{aligned} \mathfrak{A}^{I} &= A^{I}\alpha \\ \mathfrak{B}^{I} &= A^{I}\beta + B^{I}\alpha \end{aligned}$$

porroque:

$$\begin{aligned} A^{II}\gamma + B^{II}\beta + C^{II}\alpha &= A^{III} \\ B^{II}\gamma + C^{II}\beta + D^{II}\alpha &= B^{III} \\ C^{II}\gamma + D^{II}\beta + E^{II}\alpha &= C^{III} \\ &\&c. \end{aligned} \qquad \begin{aligned} \mathfrak{A}^{II} &= A^{II}\alpha \\ \mathfrak{B}^{II} &= A^{II}\beta + B^{II}\alpha \end{aligned}$$

ſique ulterius valores ſimiles inveſtigando erit: $s =$

$$\frac{A\alpha + (A\beta + B\alpha)x}{\alpha + \beta x + \gamma xx} + \frac{[A^{I}\alpha + (A^{I}\beta + B^{I}\alpha)x]x^2}{(\alpha + \beta x + \gamma xx)^2} + \frac{[A^{II} + (A^{II}\beta + B^{II}\alpha)x]x^4}{(\alpha + \beta x + \gamma xx)^3} + \&c.$$

211. Si ponatur $x = 1$, qua poſitione amplitudini nihil decedit, cum α, β, γ pro lubitu accipi poſſint, fueritque

$$s = A + B + C + D + E + F + G + \&c.$$

Computentur ſucceſſive ſequentes valores:

$$\begin{array}{l|l} A\gamma + B\beta + C\alpha = A' & A'\gamma + B'\beta + C'\alpha = A'' \\ B\gamma + C\beta + D\alpha = B' & B'\gamma + C'\beta + D'\alpha = B'' \\ C\gamma + D\beta + E\alpha = C' & C'\gamma + D'\beta + E'\alpha = C'' \\ \&c. & \&c. \end{array}$$

ſicque porro

inſuper vero brevitatis ergo ponatur: $\alpha + \beta + \gamma = m$ obtinebitur ſumma ſeriei propoſitae hoc modo expreſſa

$$s = \begin{cases} (\alpha + \beta)\left(\frac{A}{m} + \frac{A'}{m^2} + \frac{A''}{m^3} + \frac{A'''}{m^4} + \&c.\right) \\ + \alpha\left(\frac{B}{m} + \frac{B'}{m^2} + \frac{B''}{m^3} + \frac{B'''}{m^4} + \&c.\right) \end{cases}$$

212. Eodem modo denominatores ex pluribus terminis conſtantes accipi poſſunt; & quoniam operatio ex praecedentibus facile perſpicitur, hic tantum caſum pro quadrinomio evolvamus: Sit ergo

$$s = A + B + C + D + E + F + G + \&c.$$

Quaerantur valores ſequentes:

$$\begin{array}{l} A\delta + B\gamma + C\beta + D\alpha = A' \\ B\delta + C\gamma + D\beta + E\alpha = B' \\ C\delta + D\gamma + E\beta + F\alpha = C' \\ \&c. \\ A'\delta + B'\gamma + C'\beta + D'\alpha = A'' \\ B'\delta + C'\gamma + D'\beta + E'\alpha = B'' \\ C'\delta + D'\gamma + E'\beta + F'\alpha = C'' \\ \&c. \\ A''\delta + B''\gamma + C''\beta + D''\alpha = A''' \\ B''\delta + C''\gamma + D''\beta + E''\alpha = B''' \\ C''\delta + D''\gamma + E''\beta + F''\alpha = C''' \\ \&c. \end{array}$$

Tum vero ſit $\alpha + \beta + \gamma + \delta = m$; eritque

$$s = \begin{cases} (\alpha + \beta + \gamma)\left(\frac{A}{m} + \frac{A'}{m^2} + \frac{A''}{m^3} + \frac{A'''}{m^4} + \&c.\right) \\ (\alpha + \beta)\left(\frac{B}{m} + \frac{B'}{m^2} + \frac{B''}{m^3} + \frac{B'''}{m^4} + \&c.\right) + \alpha\left(\frac{C}{m} + \frac{C'}{m^2} + \frac{C''}{m^3} + \frac{C'''}{m^4} + \&c.\right) \end{cases}$$

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unde ſimul progreſſio, ſi adhuc plures partes denominatori m tribuantur, clariſſime perſpicitur.

213. Neque vero abſolute opus eſt, ut denominatores fractionum, ad quas ſummam ſeriei reducimus, ſint poteſtates eiuſdem formulae $\alpha + \beta x + \gamma x^2 + \&c.$ ſed haec ipſa in ſingulis terminis variari poteſt. Quo hoc clarius pateat, ſumamus primo tantum duos terminos, fingaturque ſeries

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

in hanc ſeriem fractionum converti:

$$s = \frac{\mathfrak{A}}{\alpha+\beta x} + \frac{\mathfrak{A}'x}{(\alpha+\beta x)(\alpha'+\beta' x)} + \frac{\mathfrak{A}''x^2}{(\alpha+\beta x)(\alpha'+\beta' x)(\alpha''+\beta'' x)} + \&c.$$

Multiplicetur primum utrinque per $\alpha + \beta x$, ponaturque

$$\begin{aligned} A\beta + B\alpha &= A' \\ B\beta + C\alpha &= B' \qquad \& \quad \mathfrak{A} = A\alpha \\ C\beta + D\alpha &= C' \\ \&c. \end{aligned}$$

fietque per x diviſo

$$A' + B'x + C'x^2 + D'x^3 + \&c. = \frac{\mathfrak{A}'}{\alpha'+\beta' x} + \frac{\mathfrak{A}''x}{(\alpha'+\beta' x)(\alpha''+\beta'' x)} + \&c.$$

Deinde ſimili modo multiplicando per $\alpha' + \beta' x$; tumque per $\alpha'' + \beta'' x$, & ita porro, ſi ſtatuatur:

$A'\beta' + B'\alpha' = A''$	$A''\beta'' + B''\alpha'' = A'''$	$A'''\beta''' + B'''\alpha''' = A''''$
$B'\beta' + C'\alpha' = B''$	$B''\beta'' + C''\alpha'' = B'''$	$B'''\beta''' + C'''\alpha''' = B''''$
$C'\beta' + D'\alpha' = C''$	$C''\beta'' + D''\alpha'' = C'''$	$C'''\beta''' + D'''\alpha''' = C''''$
&c.	&c.	&c.

fiet $\mathfrak{A}' = A'\alpha'$; $\mathfrak{A}'' = A''\alpha''$; $\mathfrak{A}''' = A'''\alpha'''$; &c.
atque hinc ſeries propoſita convertetur in hanc:

$$s = \frac{A\alpha}{\alpha+\beta x} + \frac{A'\alpha' x}{(\alpha+\beta x)(\alpha'+\beta' x)} + \frac{A''\alpha'' xx}{(\alpha+\beta x)(\alpha'+\beta' x)(\alpha''+\beta'' x)} \; \&c.$$

ubi valores α, β, α', β', α'', β'', α''', β''' &c. ſunt arbitrarii, quovis autem caſu ita accipi poſſunt, ut iſta nova ſeries maxime convergat.

214. Applicemus hoc quoque ad factores trinomiales, ſitque propoſita ſerie quacunque:

$s =$

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$$s = A + B + C + D + E + F + G + \&c.$$

$$\begin{array}{l|l} A\gamma + B\delta + C\alpha = A' & A'\gamma' + B'\delta' + C'\alpha' = A'' \\ B\gamma + C\delta + D\alpha = B' & B'\gamma' + C'\delta' + D'\alpha' = B'' \\ C\gamma + D\delta + E\alpha = C' & C'\gamma' + D'\delta' + E'\alpha' = C'' \\ \&c. & \&c. \\ A''\gamma'' + B''\delta'' + C''\alpha'' = A''' & A'''\gamma''' + B'''\delta''' + C'''\alpha''' = A'''' \\ B''\gamma'' + C''\delta'' + D''\alpha'' = B''' & B'''\gamma''' + C'''\delta''' + D'''\alpha''' = B'''' \\ C''\gamma'' + D''\delta'' + E''\alpha'' = C''' & C'''\gamma''' + D'''\delta''' + E'''\alpha''' = C'''' \\ \&c. & \&c. \end{array}$$

Deinde ſtatuatur brevitatis gratia:

$$\begin{array}{l} \alpha + \delta + \gamma = m \\ \alpha' + \delta' + \gamma' = m' \\ \alpha'' + \delta'' + \gamma'' = m'' \\ \alpha''' + \delta''' + \gamma''' = m''' \qquad \&c. \end{array}$$

eritque ſeriei propoſitae ſumma:

$$s = \frac{\alpha(A+B)}{m} + \frac{\alpha'(A'+B')}{mm'} + \frac{\alpha''(A''+B'')}{mm'm''} + \frac{\alpha'''(A'''+B''')}{mm'm''m'''} + \&c.$$

$$+ \frac{\delta A}{m} + \frac{\delta' A'}{mm'} + \frac{\delta'' A''}{mm'm''} + \frac{\delta''' A'''}{mm'm''m'''} + \&c.$$

215. Quoniam haec tam late patent, ut uſus minus clare percipi poſſit, reſtringamus transformationem §. 213. traditam, ſitque $x = -1$, ut habeatur haec ſeries:

$s = A - B + C - D + E - F + G - \&c.$ ſtatuaturque:

$$\begin{array}{l|l|l|l} B - A = A' & B' - 2A' = A'' & B'' - 3A'' = A''' & B''' - 4A''' = A'''' \\ C - B = B' & C' - 2B' = B'' & C'' - 3B'' = B''' & C''' - 4B''' = B'''' \\ D - C = C' & D' - 2C' = C'' & D'' - 3C'' = C''' & D''' - 4C''' = C'''' \\ E - D = D' & E' - 2D' = D'' & E'' - 3D'' = D''' & E''' - 4D''' = D'''' \\ \&c. & \&c. & \&c. & \&c. \end{array}$$

Quibus valoribus inventis, erit ſumma ſeriei propoſitae aequalis ſequenti ſeriei:

$$s = \frac{A}{2} - \frac{A'}{2.3} + \frac{A''}{2.3.4} - \frac{A'''}{2.3.4.5} + \frac{A''''}{2.3.4.5.6} - \&c.$$

Simili igitur modo ſeries quaecunque propoſita in innumera-

 biles

biles alias ſibi aequales tranſmutari poteſt, inter quas ſine dubio ſeries maxime convergentes reperientur, quarum ope ſumma propoſita vero proxime indagari queat.

216. Revertamur autem ad inventionem ſerierum, quarum progreſſionis legem calculus differentialis declarat. Cum igitur hoc in quantitatibus algebraicis iam ſit praeſtitum, progrediamur ad tranſcendentes, quaeraturque ſeries huic logarithmo aequalis:

$$s = l(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.)$$

fingatur quaeſito ſatisfacere haec ſeries:

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{F}x^6 + \&c.$$

Cum igitur ex illius aequationis differentiatione ſequatur

$$\frac{ds}{dx} = \frac{\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \&c.}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.} \quad \text{erit:}$$

$$(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \&c.)\frac{ds}{dx} = \alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + \&c.$$

Quia vero ex ficta aequatione eſt:

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \&c.$$

facta hac ſubſtitutione oritur haec aequatio:

$$\begin{array}{llllll}
\mathfrak{A} & + 2\mathfrak{B}x & + 3\mathfrak{C}x^2 & + 4\mathfrak{D}x^3 & + 5\mathfrak{E}x^4 & + \&c. \\
 & + \mathfrak{A}\alpha & + 2\mathfrak{B}\alpha & + 3\mathfrak{C}\alpha & + 4\mathfrak{D}\alpha & + \&c. \\
 & & + \mathfrak{A}\beta & + 2\mathfrak{B}\beta & + 3\mathfrak{C}\beta & + \&c. \\
 & & & + \mathfrak{A}\gamma & + 2\mathfrak{B}\gamma & + \&c. \\
 & & & & + \mathfrak{A}\delta & + \&c. = \\
\hline
\alpha & + 2\beta x & + 3\gamma x^2 & + 4\delta x^3 & + 5\varepsilon x^4 & + \&c.
\end{array}$$

Ex qua ſequentes determinationes obtinentur:

$$\begin{aligned}
\mathfrak{A} &= \alpha \\
\mathfrak{B} &= -\tfrac{1}{2}\mathfrak{A}\alpha + \beta \\
\mathfrak{C} &= -\tfrac{2}{3}\mathfrak{B}\alpha - \tfrac{1}{3}\mathfrak{A}\beta + \gamma \\
\mathfrak{D} &= -\tfrac{3}{4}\mathfrak{C}\alpha - \tfrac{2}{4}\mathfrak{B}\beta - \tfrac{1}{4}\mathfrak{A}\gamma + \delta \\
\mathfrak{E} &= -\tfrac{4}{5}\mathfrak{D}\alpha - \tfrac{3}{5}\mathfrak{C}\beta - \tfrac{2}{5}\mathfrak{B}\gamma - \tfrac{1}{5}\mathfrak{A}\delta + \varepsilon \qquad \&c.
\end{aligned}$$

217. Propoſita nunc ſit quantitas exponentialis:

$s =$

$$s = e^{\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.}$$

in qua e denotet numerum, cuius logarithmus hyperbolicus eſt $= 1$, atque fingatur ſeries quaeſita:

$$s = 1 + \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \&c.$$

iam enim ex caſu $x = 0$ patet, primum terminum eſſe debere unitatem. Cum igitur ſumendis logarithmis ſit

$$ls = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \&c.$$

erit differentialibus ſumtis:

$$\frac{ds}{dx} = s(\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \&c.)$$

At vero ex aequatione ficta erit:

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \&c. =$$

$$\begin{array}{llllll}
\alpha + \mathfrak{A}\alpha x + & \mathfrak{B}\alpha x^2 + & \mathfrak{C}\alpha x^3 + & \mathfrak{D}\alpha x^4 & + \&c. \\
+ 2\beta + & 2\mathfrak{A}\beta & + 2\mathfrak{B}\beta & + 2\mathfrak{C}\beta & + \&c. \\
+ & 3\gamma & + 3\mathfrak{A}\gamma & + 3\mathfrak{B}\gamma & + \&c. \\
 & & + 4\delta & + 4\mathfrak{A}\delta & + \&c. \\
 & & & + 5\varepsilon & + \&c.
\end{array}$$

ex quibus ſequentes prodeunt litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. determinationes $\mathfrak{A} = \alpha$

$$\mathfrak{B} = \beta + \tfrac{1}{2}\mathfrak{A}\alpha$$
$$\mathfrak{C} = \gamma + \tfrac{2}{3}\mathfrak{A}\beta + \tfrac{1}{3}\mathfrak{B}\alpha$$
$$\mathfrak{D} = \delta + \tfrac{3}{4}\mathfrak{A}\gamma + \tfrac{2}{4}\mathfrak{B}\beta + \tfrac{1}{4}\mathfrak{C}\alpha$$
$$\mathfrak{E} = \varepsilon + \tfrac{4}{5}\mathfrak{A}\delta + \tfrac{3}{5}\mathfrak{B}\gamma + \tfrac{2}{5}\mathfrak{C}\beta + \tfrac{1}{5}\mathfrak{D}\alpha \qquad \&c.$$

218. Si quoque arcus, cuius ſinus vel coſinus quaeritur, exprimatur binomio vel polynomio, vel etiam ſerie infinita, hoc modo quoque eius ſinus & coſinus per ſeriem infinitam exprimi poſſunt. At vero quo hoc commodiſſime fiat, non ſufficit ad differentialia prima proceſſiſſe, ſed opus eſt ut differentialia ſecundi gradus in ſubſidium vocemus. Sit igitur

$$s = \sin(\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.)$$

fingaturque ſeries quae quaeritur:

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \&c.$$

primum enim terminum conſtat evaneſcere: quia vero ad differentialia ſecunda deſcendendum eſt, coefficientem $\mathfrak{A}$ quo-

que aliunde definiri oportet, quod fiet ſi α ponamus infinite parvum. Tum enim ob arcum $= \alpha x$ ſinus ipſi fiet aequalis, eritque ergo $\mathfrak{A} = \alpha$. Ponamus nunc brevitatis gratia $z = \alpha x + \beta x^2 + \gamma x^3 + \&c.$ ut ſit $s = \sin z$, erit differentiando $ds = dz \cos z$ denuoque differentiando $dds = ddz \cos z - dz^2 \sin z$.

Quia igitur eſt $\sin z = s$ & $\cos z = \frac{ds}{dz}$; erit

$$dds = \frac{dsddz}{dz} - sdz^2, \quad \text{ſeu} \quad dzdds + sdz^3 = dsddz.$$

219. Ponamus arcum z tantum binomio exprimi eſſeque

$$z = \alpha x + \beta x^2; \quad \text{erit} \quad dz = (\alpha + 2\beta x)\, dx,$$

& poſito dx conſtante, $ddz = 2\beta dx^2$; atque $dz^3 = (\alpha^3 + 6\alpha^2 \beta x + 12\alpha\beta^2 x^2 + 8\beta^3 x^3)\, dx^3$. Deinde ob

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \&c.$$

erit

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + \&c.$$

$$\& \quad \frac{dds}{dx^2} = 2\mathfrak{B} + 6\mathfrak{C}x + 12\mathfrak{D}x^2 + \&c.$$

Quibus valoribus in aequatione differentio-differentiali ſubſtitutis fiet:

$$\begin{aligned}
\frac{dzdds}{dx^3} &= 1.2\mathfrak{B}\alpha + 2.3\mathfrak{C}\alpha x + 3.4\mathfrak{D}\alpha x^2 + 4.5\mathfrak{E}\alpha x^3 + \&c.\\
& + 2.1.2\mathfrak{B}\beta + 2.2.3\mathfrak{C}\beta + 2.3.4\mathfrak{D}\beta + \&c.\\
+\frac{sdz^3}{dx^3} &= + \mathfrak{A}\alpha^3 + \mathfrak{B}\alpha^3 + \mathfrak{C}\alpha^3 + \&c.\\
& + 6\mathfrak{A}\alpha^2\beta + 6\mathfrak{B}\alpha^2\beta + \&c.\\
& + 12\mathfrak{A}\alpha\beta^2 + \&c.\\
\hline
\frac{dsddz}{dx^3} &= 2\mathfrak{A}\beta + 4\mathfrak{B}\beta + 6\mathfrak{C}\beta + 8\mathfrak{D}\beta + \&c.
\end{aligned}$$

Unde coefficientes ſequenti modo definientur:

$\mathfrak{B} =$

$$\mathfrak{B} = \frac{2\mathfrak{A}\beta}{2\alpha}$$

$$\mathfrak{C} = 0 - \frac{\mathfrak{A}\alpha^2}{2.3}$$

$$\mathfrak{D} = -\frac{2\mathfrak{C}\beta}{4\alpha} - \frac{6\mathfrak{A}\alpha\beta}{3 \cdot 4} - \frac{\mathfrak{B}\alpha^2}{3 \cdot 4}$$

$$\mathfrak{E} = -\frac{4\mathfrak{D}\beta}{5\alpha} - \frac{12\mathfrak{A}\beta^3}{4 \cdot 5} - \frac{6\mathfrak{B}\alpha\beta}{4 \cdot 5} - \frac{\mathfrak{C}\alpha^2}{4 \cdot 5}.$$

$$\mathfrak{F} = -\frac{6\mathfrak{E}\beta}{6\alpha} - \frac{8\mathfrak{A}\beta^3}{5 \cdot 6\alpha} - \frac{12\mathfrak{B}\beta\beta}{5 \cdot 6} - \frac{6\mathfrak{C}\alpha\beta}{5 \cdot 6} - \frac{\mathfrak{D}\alpha^2}{5 \cdot 6}$$

$$\mathfrak{G} = -\frac{8\mathfrak{F}\beta}{7\alpha} - \frac{8\mathfrak{B}\beta^3}{6.7\alpha} - \frac{12\mathfrak{C}\beta\beta}{6.7} - \frac{6\mathfrak{D}\alpha\beta}{6.7} - \frac{\mathfrak{E}\alpha^2}{6.7}$$

&c. Quibus valoribus inventis erit:

$\text{ſin}(\alpha x + \beta x^2) = \mathfrak{A}x + \mathfrak{B}x + \mathfrak{C}x^3 + \mathfrak{D}x^4 +$ &c. exiſtente $\mathfrak{A} = \alpha$

220. Pari modo coſinus cuiuſque anguli in ſeriem convertitur, quia autem arcus rariſſime per polynomium exprimitur, oſtendamus uſum differentio-differentialium in invenienda ſerie pro coſinu arcus x. Sit ergo $s = \text{coſ}\,x$, & fingatur:

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \mathfrak{D}x^8 - \text{\&c.}$$

Quia eſt $ds = -dx\ \text{ſin}\,x$ & $dds = -dx^2\,\text{coſ}\,x = -sdx^2$, erit $dds + sdx^2 = 0$; ſubſtitutione ergo facta fiet:

$$\frac{dds}{dx^2} = -1.2\,\mathfrak{A} + 3.4\,\mathfrak{B}x^2 - 5.6\,\mathfrak{C}x^4 + 7.8\,\mathfrak{D}x^6 - \text{\&c.}$$

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \text{\&c.}$$

& ex coaequatione terminorum ſequitur:

$$\mathfrak{A} = \frac{1}{1.2}\ ; \quad \mathfrak{B} = \frac{\mathfrak{A}}{3.4} = \frac{1}{1.2.3.4}$$

$$\mathfrak{C} = \frac{\mathfrak{B}}{5.6} = \frac{1}{1.2.3\ldots6}\ ; \quad \mathfrak{D} = \frac{\mathfrak{C}}{7.8} = \frac{1}{1.2.3\ldots8}\ ; \text{ \&c.}$$

Patet ergo quod iam ſupra ſuſius demonſtravimus eſſe:

$$\text{coſ}\,x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3\ldots6} + \frac{x^8}{1.2.3\ldots8} - \text{\&c.}$$

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prior vero ſeries pro ſinu poſito $\beta = 0$ & $\alpha = 1$ dabit:

$$\sin x = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3\ldots 7} + \frac{x^9}{1.2.3\ldots 9} - \&c.$$

221. Ex his ſeriebus pro ſinu & coſinu notiſſimis deducuntur ſeries pro tangente, cotangente, ſecante & coſecante cuiuſvis anguli. Tangens enim prodit ſi ſinus per coſinum, cotangens ſi coſinus per ſinum, ſecans ſi radius 1 per coſinum, & coſecans ſi radius per ſinum dividatur. Series autem ex his diviſionibus ortae maxime videntur irregulares; verum excepta ſerie ſecantem exhibente reliquae per numeros Bernoullianos ſupra definitos $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. ad facilem progreſſionis legem reduci poſſunt. Quoniam enim ſupra §. 127. invenimus eſſe:

$$\frac{\mathfrak{A}u^2}{1.2} + \frac{\mathfrak{B}u^4}{1.2.3.4} + \frac{\mathfrak{C}u^6}{1.2.3\ldots 6} + \frac{\mathfrak{D}u^8}{1.2.3\ldots 8} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

erit poſito $\frac{1}{2}u = x$;

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2.3\ldots 6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots 8} - \&c.$$

atque ſi ponatur $\frac{1}{2}x$ pro x, erit:

$$\cot\tfrac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1.2} - \frac{2\mathfrak{B}x^3}{1.2.3.4} - \frac{2\mathfrak{C}x^5}{1.2.3\ldots 6} - \frac{2\mathfrak{D}x^7}{1.2.3\ldots 8} - \&c.$$

222. Hinc autem tangens cuiuſvis arcus ſequenti modo per ſeriem exprimetur. Cum ſit $\tan 2x = \frac{2\tan x}{1 - \tan x^2}$, erit:

$$\cot 2x = \frac{1}{2\tan x} - \frac{\tan x}{2} = \tfrac{1}{2}\cot x - \tfrac{1}{2}\tan x:$$

ideoque $\tan x = \cot x - 2\cot 2x$. Cum igitur ſit

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2\ldots 6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots 8} - \&c.$$

$$2\cot 2x = \frac{1}{x} - \frac{2^4\mathfrak{A}x}{1.2} - \frac{2^8\mathfrak{B}x^3}{1.2.3.4} - \frac{2^{12}\mathfrak{C}x^5}{1.2\ldots 6} - \frac{2^{16}\mathfrak{D}x^7}{1.2\ldots 8} - \&c.$$

erit hanc ſeriem ab illa ſubtrahendo:

$$\operatorname{tg} x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots 6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots 8} + \&c.$$

Si ergo hic introducantur numeri A, B, C, &c. §. 182. inventi, erit: $\tang x = \frac{2Ax}{1.2} + \frac{2^3Bx^3}{1.2.3.4} + \frac{2^5Cx^5}{1.2...6} + \frac{2^7Dx^7}{1.2...8} + \&c.$

223. Coſecans autem ſequenti modo invenietur. Quia eſt $\cot x = \tang x + 2\cot 2x = \frac{1}{\cot x} + 2\cot 2x$; erit $\cot x^2 = 2\cot x.\cot 2x + 1$, & radice extracta: $\cot x = \cot 2x + \operatorname{coſec} 2x$, unde fit $\operatorname{coſec} 2x = \cot x - \cot 2x$, & x pro $2x$, poſito $\operatorname{coſec} x = \cot\frac{1}{2}x - \cot x$. Quare cum cotangentes habeamus ſcilicet:

$$\cot\tfrac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1.2} - \frac{2\mathfrak{B}x^3}{1.2.3.4} - \frac{2\mathfrak{C}x^5}{1.2...6} - \&c.$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2...6} - \&c.$$

erit hac ſerie ab illa ſubtracta:

$$\operatorname{coſec} x = \frac{1}{x} + \frac{2(2-1)\mathfrak{A}x}{1.2} + \frac{2(2^3-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2(2^5-1)\mathfrak{C}x^5}{1.2....6} + \&c.$$

224. Per hos autem numeros Bernoullianos ſecans exprimi non poteſt, ſed requirit alios numeros, qui in ſummas poteſtatum reciprocarum imparium ingrediuntur. Si enim ponatur:

$$1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \&c. = \alpha.\frac{\pi}{2^2}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} - \&c. = \frac{\beta}{1.2}.\frac{\pi^3}{2^4}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} - \&c. = \frac{\gamma}{1.2.3.4}.\frac{\pi^5}{2^6}$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{9^7} - \&c. = \frac{\delta}{1.2...6}.\frac{\pi^7}{2^8}$$

$$1 - \frac{1}{3^9} + \frac{1}{5^9} + \frac{1}{7^9} + \frac{1}{9^9} - \&c. = \frac{\varepsilon}{1.2...8}.\frac{\pi^9}{2^{10}}$$

$$1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} + \frac{1}{7^{11}} + \frac{1}{9^{11}} - \&c. = \frac{\zeta}{1.2...10}.\frac{\pi^{11}}{2^{12}}$$

&c.

erit:

erit:

$\alpha = 1$	$\eta = 2702765$
$\beta = 1$	$\theta = 199360981$
$\gamma = 5$	$\iota = 19391512145$
$\delta = 61$	$\varkappa = 2404879661671$
$\varepsilon = 1385$	&c.
$\zeta = 50521$	

ex hiſque valoribus obtinebitur:

$$\text{ſec}\, x = \alpha + \frac{\beta}{1.2} xx + \frac{\gamma}{1.2.3.4} x^4 + \frac{\delta}{1.2 \ldots 6} x^6 + \frac{\varepsilon}{1.2 \ldots 8} x^8 + \&c.$$

225. Ut autem nexum huius ſeriei cum numeris α, β, γ, δ, &c. oſtendamus, conſideremus ſeriem ſupra tractatam:

$$\frac{\pi}{n \,\text{ſin}\, \frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

Ponatur $m = \frac{1}{2} n - k$, eritque

$$\frac{\pi}{2n \,\text{coſ}\, \frac{k}{n}\pi} = \frac{1}{n-2k} + \frac{1}{n+2k} - \frac{1}{3n-2k} - \frac{1}{3n+2k} + \frac{1}{5n-2k} + \&c.$$

Sit $\frac{k\pi}{n} = x$, ſeu $k\pi = nx$, erit

$$\frac{\pi}{2n}\,\text{ſec}\, x = \frac{\pi}{n\pi - 2nx} + \frac{\pi}{n\pi + 2nx} - \frac{\pi}{3n\pi - 2nx} - \&c. \quad \text{ſeu}$$

$$\text{ſec}\, x = \frac{2}{\pi - 2x} + \frac{2}{\pi + 2x} - \frac{2}{3\pi - 2x} - \frac{2}{3\pi + 2x} + \frac{2}{5\pi - 2x} + \&c.$$

$$\text{ſec}\, x = \frac{4\pi}{\pi^2 - 4x^2} - \frac{4 \cdot 3\pi}{9\pi^2 - 4xx} + \frac{4 \cdot 5\pi}{25\pi^2 - 4xx} - \frac{4 \cdot 7\pi}{49\pi^2 - 4xx} + \&c.$$

ſi nunc ſinguli termini in ſeries convertantur, fiet:

$$\text{ſec}\, x = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.\right)$$
$$+ \frac{2^4 x^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.\right)$$
$$+ \frac{2^6 x^4}{\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c.\right) \quad \&c.$$

qua-

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quarum ſerierum loco ſi valores ſuprà aſſignati ſubſtituantur, prodibit eadem ſeries pro ſecante, quam dedimus.

226. Hinc ſimul patet lex, qua numeri α, β, γ, &c. quibus ſummae poteſtatum imparium conſtituuntur, procedunt. Cum enim ſit $\sec. x = \frac{1}{\cos x} = \alpha + \frac{\beta}{1.2}x^2 + \frac{\gamma}{1.2.3.4}x^4 + \frac{\delta}{1.2\ldots 6}x^6 + \&c.$ neceſſe eſt ut haec ſeries aequalis ſit fractioni

$$\frac{1}{1 - \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2\ldots 6} + \frac{x^8}{1.2\ldots 8} - \&c.}$$

aequalitate ergo conſtituta fiet $1 =$

$$\begin{array}{lllll}
\alpha + \frac{\beta}{1.2}x^2 & + \frac{\gamma}{1.2.3.4}x^4 & + \frac{\delta}{1.2\ldots 6}x^6 & + \frac{\varepsilon}{1.2\ldots 8}x^8 & + \&c. \\
\quad - \frac{\alpha}{1.2} & - \frac{\beta}{1.2.1.2} & - \frac{\gamma}{1.2.1\ldots 4} & - \frac{\delta}{1.2.1\ldots 6} & \&c. \\
 & + \frac{\alpha}{1.2.3.4} & + \frac{\beta}{1\ldots 4.1.2} & + \frac{\gamma}{1\ldots 4.1\ldots 4} & \&c. \\
 & & - \frac{\alpha}{1\ldots 6} & - \frac{\beta}{1\ldots 6.1.2} & \&c. \\
 & & & + \frac{\alpha}{1\ldots 8} & \&c.
\end{array}$$

unde ſequuntur hae aequationes:

$\alpha = 1$; $\beta = \frac{2.1}{1.2}\alpha$;

$= \frac{4.3}{1.2}\beta - \frac{4.3.2.1}{1.2.3.4}\alpha$; $\delta = \frac{6.5}{1.2}\gamma - \frac{6.5.4.3}{1.2.3.4}\beta + \frac{6\ldots 1}{1\ldots 6}\alpha$;

$\varepsilon = \frac{8.7}{1.2}\delta - \frac{8.7.6.5}{1.2.3.4}\gamma + \frac{8\ldots 3}{1\ldots 6}\beta - \frac{8\ldots 1}{1\ldots 8}\alpha$; &c.

Ex hiſque formulis inventi ſunt iſtarum litterarum valores, quos in §. 224. exhibuimus; & quorum ope ſummae ſerierum in hac forma contentarum, $1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \&c.$ ſi n fuerit numerus impar, exprimi poſſunt.

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CAPUT IX.

DE USU CALCULI DIFFERENTIALIS IN AEQUATIONIBUS RESOLVENDIS.

227.

Conſtitutionem aequationum ad functionum rationem reduci poſſe ſupra iam ſatis oſtenſum eſt. Denotet enim y functionem quamcunque ipſius x, ſi ponatur $y = 0$, in hac forma omnes omnino aequationes finitae ſive ſint algebraicae ſive tranſcendentes comprehenduntur. Aequatio autem $y = 0$ reſolvi dicitur, ſi is ipſius x valor definiatur qui in functione y ſubſtitutus, eam actu nihilo aequalem reddat. Plerumque autem plures eiuſmodi valores pro x dantur, qui aequationis $y = 0$ radices vocantur. Si igitur ponamus numeros f, g, h, i, &c. eſſe radices aequationis $y = 0$, functio y ita erit comparata, ut ſi in ea loco x vel f, vel g, vel h, &c. ſubſtituatur, fiat revera $y = 0$.

228. Quoniam igitur functio y evaneſcit, ſi in ea loco x ponatur f, ſeu $x + (f - x)$, exiſtente f radice aequationis $y = 0$, erit per ea, quae ſupra de functionibus demonſtravimus:

$$0 = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \&c.$$

ex qua aequatione valor radicis f ita definitur, ut quicquid pro x fuerit poſitum, indeque valores quantitatum y, $\frac{dy}{dx}$, $\frac{ddy}{2dx^2}$, &c. ſubſtituti, ſemper reſultet aequatio verum valorem ipſius f exhibens. Quo hoc clarius percipiatur, ponamus eſſe $y = x^3 - 2x^2 + 3x - 4$; erit

$$\frac{dy}{dx} = 3xx - 4x + 3; \quad \frac{ddy}{2dx^2} = 3x - 2; \quad \& \quad \frac{d^3y}{6dx^3} = 1.$$

Quibus valoribus ſubſtitutis oritur:

$$o = x^3 - 2x^2 + 3x - 4 + (f - x)(3xx - 4x + 3) + (f - x)^2(3x - 2) + (f - x)^3$$

ſeu multiplicationibus actu inſtitutis:

$$f^3 - 2ff + 3f - 4 = o$$

oritur ſcilicet aequatio ſimilis ipſi propoſitae, quae propterea eaſdem continet radices.

229. Quanquam autem hoc modo ad novam aequationem non pervenitur, ex qua valor radicis f facilius definiri queat; tamen hinc ingentia ſubſidia ad inventionem radicum deduci poſſunt. Si enim pro x aſſumtus fuerit valor iam proxime ad quampiam radicem aequationis accedens, ita ut $f - x$ ſit quantitas valde parva, tum termini aequationis:

$$o = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \&c.$$

vehementer convergent, hancque ob cauſam non multum a veritate aberrabitur, ſi praeter binos terminos initiales reliqui reiiciantur. Erit ergo ſi pro x iam valor cuipiam aequationis $y = o$ radici prope aequalis fuerit aſſumtus, proxime

$$o = y + \frac{(f-x)dy}{dx} \text{ ſeu } f = x - \frac{ydx}{dy},$$

ex qua formula etſi non verus, tamen admodum propinquus radicis f valor reperietur, qui deinceps denuo loco x ſubſtitutus, multo adhuc propiorem valorem pro f ſuppeditabit; ſicque continuo propius ad verum radicis f valorem accedetur.

230. Hinc igitur primum radices omnium dignitatum ex quibuſcunque numeris extrahi poſſunt. Sit enim propoſitus numerus $a^n + b$ ex quo radicem poteſtatis n extrahi oporteat. Ponatur $x^n = a^n + b$ ſeu $x^n - a^n - b = o$, ut ſit $y = x^n - a^n - b$; erit

$$\frac{dy}{dx} = nx^{n-1}; \quad \frac{ddy}{2dx^2} = \frac{n(n-1)}{1 \cdot 2}x^{n-2}; \quad \frac{d^3y}{6dx^3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3};$$

&c.

 Hinc

Hinc ſi radix quaeſita ponatur $=f$, ut ſit $f=\sqrt[n]{(a^n+b)}$ erit:

$$0=x^n-a^n-b+n(f-x)x^{n-1}+\frac{n(n-1)}{1.2}(f-x)^2x^{n-2}+\text{\&c.}$$

Si igitur pro x iam ſtatuatur numerus ad valorem radicis quaeſitae f prope accedens, quod fiet ponendo $x=a$, ſi quidem b ſit numerus tam parvus, ut $a^n+b<(a+1)^n$; erit $b=na^{n-1}(f-a)$ proxime, ideoque $f=a+\frac{b}{na^{n-1}}$, unde valor radicis multo propius cognoſcetur. Sin autem adhuc tertium terminum aſſumere velimus, ut ſit

$$b=na^{n-1}(f-a)+\frac{n(n-1)}{1.2}a^{n-2}(f-a)^2; \text{ fiet}$$

$$(f-a)^2=-\frac{2a}{n-1}(f-a)+\frac{2b}{n(n-1)a^{n-2}}, \text{ ideoque}$$

$$f=a-\frac{a}{n-1}\pm\sqrt{\left(\frac{aa}{(n-1)^2}+\frac{2b}{n(n-1)a^{n-2}}\right)} \text{ ſeu}$$

$$f=\frac{(n-2)a+\sqrt{[aa+2(n-1)b:na^{n-2}]}}{n-1}.$$

Quare ope extractionis radicis quadratae valor radicis f adhuc propius reperietur.

EXEMPLUM.

Quaeramus radicem quadratam ex numero quocunque c, *ſeu ſit* $xx-c=y$.

Ponatur ergo numerus radici proximus $=a$, & $b=c-aa$, ob $aa+b=c$, & quia eſt $n=2$, fiet prior formula $f=a+\frac{c-aa}{2a}=\frac{c+aa}{2a}$; altera vero dat $f=\sqrt{c}$, quae eſt ipſa radix quaeſita. Cum igitur ſit proxime radix $=\frac{c+aa}{2a}$, hic ipſe valor pro a ſcribatur, eritque propius radix $f=\frac{cc+6aac+a^4}{4a(c+aa)}$. Sit verbi gratia $c=5$; erit ex priori

formula $f = \frac{5}{2a} + \frac{a}{2}$. Ponatur ergo $a = 2$; erit $f = 2,25$; nunc ponatur $a = 2,25$; fiet $f = 2,236111$, ſtatuatur porro $a = 2,236111$, erit $f = 2,2360679$, qui valor iam minime a vero diſcrepat.

231. Simili autem modo radix cuiuſcunque aequationis inveniri poteſt proxime ope aequationis $f = x - \frac{ydx}{dy}$, poſtquam ſcilicet pro x aſſumtus fuerit valor parum a quapiam aequationis radice diſcrepans. Ad huiuſmodi vero valorem pro x inveniendum, ſubſtituantur ſucceſſive pro x varii valores, inter eoſque is eligatur, qui functionis y minimum hoc eſt cyphrae proximum valorem indicat. Sic ſi ſit $y = x^3 - 2xx + 3x - 4$,

poſito $x = 0$ fit $y = -4$
$x = 1$ . . . $y = -2$
$x = 2$ . . . $y = +2$

unde videmus radicem contineri inter valores 1 & 2, ipſius x. Cum ergo ſit $\frac{dy}{dx} = 3xx - 4x + 3$, habebitur pro radice f aequationis $x^3 - 2xx + 3x - 4 = 0$ invenienda haec aequatio:

$$f = x - \frac{ydx}{dy} = x - \frac{x^3 - 2xx + 3x - 4}{3xx - 4x + 3}.$$

Sit ergo $x = 1$; fiet $f = 1 + \frac{2}{2} = 2$. Nunc ponatur $x = 2$, fiet $f = 2 - \frac{2}{7} = \frac{12}{7}$. Sit ergo $x = \frac{12}{7}$; erit $f = \frac{12}{7} - \frac{104}{1701} = \frac{2812}{1701} = 1,653$. Si ulterius progredi velimus, logarithmis commodius utemur.

Ponatur ergo $x = 1,653$, eritque

lx

$lx = 0,2182729$	$x = 1,653000$
$lx^2 = 0,4365458$	$x^2 = 2,732409$
$lx^3 = 0,6548187$	$x^3 = 4,516673$
$x^3 = 4,516673$	
$3x = 4,959000$	
$9,475673$	$3xx + 3 = 11,197227$
$2xx + 4 = 9,464818$	$4x = 6,612000$
num. $0,010855$	den. $= 4,585227$
l num. $= 8,0356298$	
l den. $= 0,6613608$	$x = 1,653000$
l fract. $= 7,3742690$;	fractio $= 0,002367$
	$f = 2,650633$

qui valor iam proxime ad verum accedit.

232. Citiores autem approximationes ex generali expreſſione deducere poterimus. Cum enim poſita functione quacunque $y = 0$, ſi radix huius aequationis fuerit $x = f$, invenerimus eſſe:

$$0 = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \&c.$$

ſit $f - x = z$, ita ut ſit radix $f = x + z$, atque ponatur

$$\frac{dy}{dx} = p;\quad \frac{dp}{dx} = q;\quad \frac{dq}{dx} = r;\quad \frac{dr}{dx} = s;\quad \text{erit:}$$

$$0 = y + zp + \frac{z^2 q}{2} + \frac{z^3 r}{6} + \frac{z^4 s}{24} + \frac{z^5 t}{120} + \&c.$$

in qua aequatione ſumto pro x valore quocunque, ex quo ſimul y, p, q, r, s, &c. determinantur, inveniri debet quantitas z, qua inventa habebitur aequationis propoſitae $y = 0$, radix $f = x + z$. In id ergo eſt incumbendum, ut quam commodiſſime ex hac aequatione valor incognitae z eruatur.

233. Fingatur pro z ſeries convergens haec:

$$z = A + B + C + D + E + \&c.$$

atque facta ſubſtitutione erit:

$$\begin{aligned}
y &= y\\
pz &= Ap + Bp + Cp + Dp + Ep + \&c.\\
\tfrac{1}{2}qz^2 &= \quad + \tfrac{1}{2}A^2q + ABq + ACq + ADq + \&c.\\
&\qquad\qquad + \tfrac{1}{2}BBq + BCq + \&c.\\
\tfrac{1}{6}rz^3 &= \qquad\qquad \tfrac{1}{6}A^3r + \tfrac{1}{2}A^2Br + \tfrac{1}{2}A^2Cr + \&c.\\
&\qquad\qquad\qquad + \tfrac{1}{2}AB^2r + \&c.\\
\tfrac{1}{24}sz^4 &= \qquad\qquad\qquad \tfrac{1}{24}A^4s + \tfrac{1}{6}A^3Bs + \&c.\\
\tfrac{1}{120}tz^5 &= \qquad\qquad\qquad\qquad \tfrac{1}{120}A^5t + \&c.
\end{aligned}$$

Unde obtinentur ſequentes aequationes:

$$A = -\frac{y}{p}$$

$$B = -\frac{yyq}{2p^3}$$

$$C = -\frac{y^3qq}{2p^5} + \frac{y^3r}{6p^4}$$

$$D = -\frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5}$$

&c. ideoque erit:

$$z = -\frac{y}{p} - \frac{y^2q}{2p^3} - \frac{y^3qq}{2p^5} + \frac{y^3r}{6p^4} - \frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5} - \&c.$$

EXEMPLUM.

Sit propoſita haec aequatio $x^5 + 2x - 2 = 0$.

Erit ergo $y = x^5 + 2x - 2$;

$$\frac{dy}{dx} = p = 5x^4 + 2 \quad ; \quad \frac{dp}{dx} = q = 20x^3 \quad ;$$

$$\frac{dq}{dx} = r = 60x^2 \quad ; \quad \frac{dr}{dx} = s = 120x \quad ; \quad \&c.$$

Ponatur autem nunc $x = 1$, quia hic valor parum a radice diſcrepat, erit:

$y = 1$; $p = 7$; $q = 20$; $r = 60$; $s = 120$. unde fiet:

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{200}{7^5} + \frac{10}{7^4} - \frac{5\cdot 1000}{7^7} + \frac{500}{7^6} - \frac{5}{7^5}$$

ſeu

ſeu $x = -\frac{1}{7} - \frac{10}{7^3} - \frac{130}{7^5} - \frac{1745}{7^7}$ &c. eritque ergo $x = -0,18$, & radix $f = 0,82$, qui valor ſi denuo loco x ſubſtitueretur, prodiret radix maxime verae propinqua.

234. Invenimus ergo ſeriem infinitam, quae cuiuſvis aequationis radicem exprimit: ea autem hoc laborat incommodo ut tum lex progreſſionis non pateat, tum ipſa nimis ſit perplexa atque ad uſum non ſatis accommodata. Alio igitur modo idem negotium ſuſcipiamus, ſeriemque magis regularem inveſtigemus, cuiuſcunque aequationis propoſitae radicem exprimentem.

Sit ut ante propoſita aequatio $y = 0$, exiſtente y functione quacunque ipſius x; & quaeſtio huc redit, ut valor ipſius x definiatur, qui loco x ſubſtitutus functionem y reddat nihilo aequalem. Cum autem y ſit functio ipſius x, viciſſim x tanquam functio ſpectari poterit ipſius y, atque hac conſideratione adhibita quaerendus eſt valor ipſius functionis x, quem induit, cum quantitas y evaneſcit. Si igitur f ponatur deſignare iſtum ipſius x valorem, qui erit radix aequationis $y = 0$, quoniam x abit in f, ſi ſtatuatur $y = 0$, erit per ea quae ſupra ſunt demonſtrata:

$$f = x - \frac{ydx}{dy} + \frac{y^2 ddx}{2dy^2} - \frac{y^3 d^3x}{6dy^3} + \frac{y^4 d^4x}{24dy^4} - \&c.$$

in qua aequatione ſtatuitur differentiale dy conſtans. Si igitur ponatur: $\frac{dx}{dy} = p$; $\frac{dp}{dy} = q$; $\frac{dq}{dy} = r$; $\frac{dr}{dy} = s$; &c. erit his valoribus introductis, ut conſideratio differentialis conſtantis exuatur:

$$f = x - py + \frac{1}{2}qy^2 - \frac{1}{6}ry^3 + \frac{1}{24}sy^4 - \frac{1}{120}ty^5 + \&c.$$

235. Tributo ergo ipſi x quocunque valore, ſimul valores ipſius y, atque quantitatum p, q, r, s, &c. determinabuntur; hiſque inventis habebitur ſeries infinita valorem radicis f ex-

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f exprimens. Sin autem aequatio $y = 0$ plures admittat radices, tum eae prodibunt, ſi pro x diverſi valores aſſumantur: quia enim y eundem valorem induere poteſt, etiamſi ipſi x diverſi valores tribuantur, mirum non eſt eandem ſeriem ſaepenumero plures valores ſuppeditare poſſe. Quo igitur his caſibus ambiguitas tollatur, ſimulque ſeries convergens reddatur, pro x aſſumi debet valor iam prope ad valorem eius radicis, quae quaeritur, accedens. Hoc enim modo valor ipſius y fiet admodum parvus, ſerieique termini vehementer decreſcent, ita ut paucis terminis ſumendis iam ſatis iuſtus valor pro f inveniatur. Hic igitur valor ſi deinceps loco x ſubſtituatur, quantitas y multo minor evadet, ſerieſque multo magis converget; hocque modo ſtatim radix f tam exacte innoteſcet, ut error futurus ſit minimus. Hincque ſumma huius expreſſionis praerogativa prae ea, quam ante elicueramus, manifeſto perſpicitur.

236. Ponamus extrahendam eſſe radicem poteſtatis n ex numero quocunque N. Sumta igitur proxima poteſtate exponentis n, numerus propoſitus facile reſolvetur in hanc formam $N = a^n + b$. Erit ergo

$$x^n = a^n + b \quad \& \quad y = x^n - a^n - b; \quad \text{unde fit:}$$

$$dy = nx^{n-1}\,dx \quad ; \quad \& \quad \frac{dx}{dy} = p = \frac{1}{nx^{n-1}}$$

$$dp = -\frac{(n-1)\,dx}{nx^n} \quad ; \quad \& \quad \frac{dp}{dy} = q = -\frac{n-1}{nnx^{2n-1}}$$

$$dq = \frac{(n-1)(2n-1)dx}{nnx^{2n}}\,; \& \quad \frac{dq}{dy} = r = \frac{(n-1)(2n-1)}{n^3 x^{3n-1}}$$

$$dr = -\frac{(n-1)(2n-1)(3n-1)dx}{n^3 x^{3n}}\,; \& \quad s = -\frac{(n-1)(2n-1)(3n-1)}{n^4 x^{4n-1}}$$

&c.

Ponatur nunc $x = a$, eritque $y = -b$, atque radix quaeſita $f = \sqrt[n]{(a^n + b)}$ hoc modo exprimetur:

$$f = a + \frac{b}{na^{n-1}} - \frac{(n-1)bb}{n.2n.a^{2n-1}} + \frac{(n-1)(2n-1)b^3}{n.2n.3n.a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{n.2n.3n.4n.a^{4n-1}} + \&c.$$

ſicque prodit eadem ſeries, quae vulgo per evolutionem binomii $(a^n + b)^{\frac{1}{n}}$ erui ſolet.

237. Poſtquam ergo in actuali extractione radix proxime vera a fuerit inventa, ſimulque reſiduum b fuerit repertum, tum ad radicem inſuper addi oportet valor fractionis $\frac{b}{na^{n-1}}$, quo propius vera radix obtineatur. Erit autem $a^{n-1} = \frac{N - b}{a}$, ob $N = a^n + b$. At vero hoc modo radix iuſto maior invenietur, quoniam tertius terminus ſubtrahi debet. Quo igitur per diviſionem reſidui b radix multo propius ad verum accedens inveniatur idoneus diviſor debet inveſtigari, qui fingatur eſſe $na^{n-1} + \alpha b + \beta bb + \gamma b^3 +$ &c. Cum igitur debeat eſſe:

$$\frac{b}{na^{n-1} + \alpha b + \beta b^2 + \gamma b^3 + \&c.} =$$

$$\frac{b}{na^{n-1}} - \frac{(n-1)bb}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)b^3}{6n^3a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^4a^{4n-1}} + \&c.$$

fiet multiplicatione per $na^{n-1} + \alpha b + \beta b^2 + \gamma b^3 +$ &c. inſtituta:

$$b = b - \frac{(n-1)bb}{2na^n} + \frac{(n-1)(2n-1)b^3}{6n^2a^{2n}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^3a^{3n}} + \&c.$$
$$+ \frac{\alpha b^2}{na^{n-1}} - \frac{(n-1)\alpha b^3}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)\alpha b^4}{6n^3a^{3n-1}}$$
$$+ \frac{\beta b^3}{na^{n-1}} - \frac{(n-1)\beta b^4}{2n^2a^{2n-1}}$$
$$\frac{\gamma b^4}{na^{n-1}}$$

Hinc deducuntur ſequentes determinationes:

$$\alpha = \frac{n-1}{2a}$$

$$\beta = \frac{(n-1)\alpha}{2na^{n}} - \frac{(n-1)(2n-1)}{6na^{n+1}} = -\frac{(n-1)(n+1)}{12na^{n+1}}$$

$$\gamma = \frac{(n-1)\beta}{2na^{n}} - \frac{(n-1)(2n-1)\alpha}{6nna^{2n}} + \frac{(n-1)(2n-1)(3n-1)}{24n^{2}a^{2n+1}}$$

ſeu $\gamma = \frac{(n-1)(n+1)}{24na^{2n+1}}$

Fractio ergo ad radicem iam inventam a inſuper addenda erit:

$$\frac{b}{na^{n-1} + \frac{(n-1)b}{2a} - \frac{(nn-1)bb}{12na^{n+1}} + \frac{(nn-1)b^{3}}{24na^{2n+1}}}$$

238. Quod ſi ergo radix quadrata extrahi debeat ex numero N, atque inventa iam ſit radix proxima $= a$, cum reſiduo $= b$, ad radicem inventam inſuper addi debet quotus, qui oritur, ſi reſiduum b dividatur per

$$2a + \frac{b}{2a} - \frac{bb}{8a^{3}} + \frac{b^{3}}{16a^{5}} - \&c.$$

Sin autem radix cubica extrahi debeat, tum reſiduum b dividi debet per $3a^{2} + \frac{b}{a} - \frac{2bb}{9a^{4}} + \frac{b^{3}}{9a^{7}} - \&c.$ quarum formularum uſum in his exemplis declarabimus.

EXEMPLUM I.

Extrahatur radix quadrata ex numero 200.

Ponatur $N = 200$, & cum proximum quadratum ſit 196, erit $a = 14$, & reſiduum $b = 4$, quod propterea dividi debebit per $28 + \frac{1}{7} - \frac{1}{7} \cdot \frac{1}{196} + \frac{1}{7.196.98}$, eritque ergo diviſor $= 28,142135$, per quem ſi 4 dividatur, obtinebitur fractio decimalis ad 14 addenda, quae iuſta erit ad 10 figuras & ultra.

EXEMPLUM II.

Extrahatur radix cubica ex numero $N = 10$.

Proximus cubus eſt 8, & reſiduum $= 2$, unde $a = 2$ & $b = 2$, atque diviſor $= 12 + 1 - \frac{1}{18} = 12{,}9444$. Quare radix cubica quaeſita erit proxime $= 2\frac{2}{12{,}9444} = 2\frac{10000}{64722}$.

239. Series pro radice inventa etiam conſiderari poteſt tanquam recurrens orta ex quapiam fractione, hoc enim modo plures termini ſeriei ad multo pauciores, qui numeratorem & denominatorem fractionis conſtituant, revocabuntur. Sic levi attentione adhibita perſpicietur fore proxime:

$$(a+b)^n = a^n . \frac{a + \frac{n+1}{2} b}{a + \frac{n-1}{2} b}$$

atque adhuc propius

$$(a+b)^n = a^n . \frac{aa + \frac{n+2}{2} ab + \frac{(n+1)(n+2)}{12} bb}{aa - \frac{n-2}{2} ab + \frac{(n-1)(n-2)}{12} bb}.$$

Simili modo plures terminos introducendo fractiones adhuc accuratiores obtineri poſſunt:

$$(a+b)^n = a^n . \frac{a^3 + \frac{n+3}{2} a^2 b + \frac{(n+3)(n+2)}{10} ab^2 + \frac{(n+3)(n+2)(n+1)}{120} b^3}{a^3 - \frac{n-3}{2} a^2 b + \frac{(n-3)(n-2)}{10} ab^2 - \frac{(n-3)(n-2)(n-1)}{120} b^3}$$

Quin etiam huiuſmodi forma generalis exhiberi poteſt, ad quam commode exprimendam ſit:

A =

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$$A = \frac{m(n+m)}{1 \cdot 2m} \qquad \mathfrak{A} = \frac{m(n-m)}{1 \cdot 2m}$$

$$B = \frac{(m-1)}{2}\frac{(n+m-1)}{(2m-1)} A \qquad \mathfrak{B} = \frac{(m-1)}{2}\frac{(n-m+1)}{(2m-1)} \mathfrak{A}$$

$$C = \frac{(m-2)}{3}\frac{(n+m-2)}{(2m-2)} B \qquad \mathfrak{C} = \frac{(m-2)}{3}\frac{(n-m+2)}{(2m-2)} \mathfrak{B}$$

$$D = \frac{(m-3)}{4}\frac{(n+m-3)}{(2m-3)} C \qquad \mathfrak{D} = \frac{(m-3)}{4}\frac{(n-m+3)}{(2m-3)} \mathfrak{C}$$

&c. &c.

His autem valoribus determinatis erit:

$$(a+b)^n = a^n \cdot \frac{a^m + Aa^{m-1}b + Ba^{m-2}b^2 + Ca^{m-3}b^3 + \&c.}{a^m - \mathfrak{A}a^{m-1}b + \mathfrak{B}a^{m-2}b^2 - \mathfrak{C}a^{m-3}b^3 + \&c.}$$

240. Si igitur hic pro n ſubſtituatur numerus fractus, iſtae formulae ad extractionem radicum apprime erunt accommodatae. Sic ſi radix quaecunque poteſtatis n extrahi debeat ex forma $a^n + b$, ſequentes formulae in uſum vocari poſſunt:

$$(a^n+b)^{\frac{1}{n}} = a.\frac{2na^n + (n+1)b}{2na^n + (n-1)b}$$

$$(a^n+b)^{\frac{1}{n}} = a.\frac{12n^2a^{2n} + 6n(2n+1)a^nb + (2n+1)(n+1)bb}{12n^2a^{2n} + 6n(2n-1)a^nb + (2n-1)(n-1)bb}.$$

Sin autem ponatur $a^n + b = N$, ut ſit $a^n = N - b$; erit

$$(a^n+b)^{\frac{1}{n}} = a.\frac{2nN - (n-1)b}{2nN - (n+1)b}$$

$$(a^n+b)^{\frac{1}{n}} = a.\frac{12n^2N^2 - 6n(2n-1)Nb + (2n-1)(n-1)bb}{12n^2N^2 - 6n(2n+1)Nb + (2n+1)(n+1)bb}.$$

241. Formula igitur generalis pro radice cuiuſque aequationis invenienda in aequationibus, quae ex pluribus terminis conſtant, eundem praeſtat uſum, quem ſolita regula binomii ad reſolutionem aequationum purarum $x^n = c$ afferre ſolet, atque adeo hoc caſu in regulam illam ipſam abit. Sin

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Sin autem aequatio fuerit affecta vel etiam tranſcendens, expreſſio noſtra generalis ſemper aequali ſucceſſu in uſum vocatur, ſeriemque praebet infinitam, quae valorem radicis exhibet. Quamobrem cum in hoc negotio ſumma vis iſtius formulae generalis conſiſtat, eius uſum hic aliquanto fuſius oſtendamus. Sit igitur propoſita haec aequatio affecta tribus terminis conſtans:

$$x^n + cx = N,$$

denotantibus c & N quantitates quaſcunque datas. Ponatur $x^n + cx - N = y$, erit $dy = (nx^{n-1} + c)\,dx$, hincque fiet $p = \frac{1}{nx^{n-1} + c}$, tum eſt

$$dp = -\frac{n(n-1)x^{n-2}\,dx}{(nx^{n-1}+c)^2}; \quad \& \quad q = -\frac{n(n-1)x^{n-2}}{(nx^{n-1}+c)^3}.$$

Simili modo ob $r = \frac{dq}{dy}$; $s = \frac{dr}{dy}$; &c. reperietur:

$$r = \frac{n^2(n-1)(2n-1)x^{2n-4} - n(n-1)(n-2)cx^{n-3}}{(nx^{n-1}+c)^5}$$

$$s = -\frac{n^3(n-1)(2n-1)(3n-1)x^{3n-6} + 4n^2(n-1)(n-2)(2n-1)cx^{2n-5} - n(n-1)(n-2)(n-3)c^2x^{n-4}}{(nx^{n-1}+c)^7}$$

$$t = +\frac{n^4(n-1)(2n-1)(3n-1)(4n-1)x^{4n-8} - n^3(n-1)(n-2)(2n-1)(29n-11)cx^{3n-7} + n^2(n-1)(n-2)(2n-1)(nn-29)c^2x^{2n-6} - n(n-1)(n-2)(n-3)(n-4)c^3x^{n-5}}{(nx^{n-1}+c)^9}$$

&c.

Quibus valoribus inventis, erit aequationis propoſitae radix

$$f = x - py + \frac{1}{2}qyy - \frac{1}{6}ry^3 + \frac{1}{24}sy^4 - \frac{1}{120}ty^5 + \&c.$$

quicquid enim pro x ſubſtituatur, unde ſimul litterae y, p, q, r, &c. valores determinatos induunt, ſumma ſeriei aequabitur valori unius radicis.

EXEMPLUM I.

Sit propoſita haec aequatio $x^3 + 2x = 2$.

Erit $c = 2$, $N = 2$, & $n = 3$, atque $y = x^3 + 2x - 2$.

Po-

Ponatur $x = 1$, erit $y = 1$, & $p = \frac{1}{5}$; $q = -\frac{6}{5^3}$; $r = \frac{78}{5^5}$; $s = -\frac{16.90}{5^7}$ &c. atque aequationis radix erit:

$$f = 1 - \frac{1}{5} - \frac{3}{5^3} - \frac{13}{5^5} - \frac{60}{5^7} - \text{\&c.} = 0,771072.$$

Ponatur nunc $x = 0,77$, & quia eſt $y = x^3 + 2x - 2$; $p = \frac{1}{3xx + 2}$; $q = -6p^3x$; $r = 90xxp^5 - 12p^5$; atque $s = -2160p^7x^3 + 720p^7x$; habebitur logarithmis adhibendis:

$lx = 9,8864907$ | $x = 0,77$

$lx^2 = 9,7729814$ | $x^2 = 0,5929$

$lx^3 = 9,6594721$ | $x^3 = 0,456533$

$2x = 1,54$

$x^3 + 2x = 1,996533$

Ergo $y = -0,003467$

$l-y = 7,5399538$; $3xx + 2 = 3,7787$

$lp = 9,4226575$; $l(3xx+2) = 0,5773424$

$l-py = 6,9626113$; $-py = 0,000917511$

$lp^3 = 8,2679725$

$lx = 9,8864907$

$l3 = 0,4771213$

$ly^2 = 5,0799076$

$l-\frac{1}{2}qyy = 3,7114922$ $\quad -\frac{1}{2}qyy = 0,000000514.$

Ergo radix $f = 0,770916997$, quae vix in ultima figura a vero aberrabit.

EXEMPLUM II.

Sit propoſita aequatio $x^4 - 2xx + 4x = 8.$

Ponatur $y = x^4 - 2xx + 4x - 8$, erit $dy = 4dx(x^3 - x + 1)$

$$p = \frac{1}{4(x^3 - x + 1)}; \quad \frac{dp}{dx} = \frac{-3xx + 1}{4(x^3 - x + 1)^2} \text{ Ergo}$$

$$q = \frac{-3xx + 1}{16(x^3 - x + 1)^3}; \quad \frac{dq}{dx} = \frac{21x^4 - 12xx - 6x + 3}{16(x^3 - x + 1)^4}$$

$$\& \quad r = \frac{21x^4 - 12xx - 6x + 3}{64(x^3 - x + 1)^5}; \qquad \&c.$$

ex quibus erit radix aequationis propoſitae:

$$f = x - \frac{y}{4(x^3 - x + 1)} - \frac{(3xx - 1)yy}{32(x^3 - x + 1)^3} - \frac{(7x^4 - 4xx - 2x + 1)y^3}{128(x^3 - x + 1)^5} - \&c.$$

Oportet ergo ipſi x idoneum valorem tribui, quo ſeries iſta fiat convergens. Primum autem perſpicuum eſt, ſi ipſi x tribueretur talis valor, quo fieret $x^3 - x + 1 = 0$, tum omnes ſeriei terminos praeter primum evadere infinitos, neque adeo exinde quicquam concludi poſſe. Convenit ergo ipſi x eiuſmodi valorem aſſignare, quo & y fiat exiguum & $x^3 - x + 1$ non admodum parvum. Sit $x = 1$, erit $y = -5$, & $f = 1 + \frac{5}{4} - \frac{25}{16} + \frac{125}{64} - \&c.$ ubi cum tres termini $\frac{5}{4} - \frac{25}{16} + \frac{125}{64}$ congruant cum progreſſione geometrica, cuius ſumma eſt $\frac{5}{9}$ erit circiter $f = \frac{14}{9}$. Statuamus ergo $x = \frac{3}{2}$, erit $y = -\frac{23}{16}$; & $x^3 - x + 1 = \frac{23}{8}$, unde fit:

$$f = \frac{3}{2} + \frac{1}{8} - \frac{1}{64} + \frac{391}{256.529} - \&c. = 1,61.$$

Ponatur nunc $x = 1,61$; erit:

$lx = 0,2068259$	$x = 1,61$	fit $x^3 - x + 1 = z$
$lx^2 = 0,4136518$	$x^2 = 2,5921$	
$lx^3 = 0,6204777$	$x^3 = 4,173281$	
$lx^4 = 0,8273036$	$x^4 = 6,718983$	
	hinc	
$l{-y} = 8,4016934$	$y = -0,025217$	
$lz = 0,5518502$	$z = 3,563281$	
$l\frac{-y}{z} = 7,8498432$		

$l4 =$

$l4 = 0,6020600$	
$l\frac{-y}{4z} = 7,2477832$	$\frac{-y}{4z} = 0,0017692$
$l(3xx-1) = 0,8309926$	$3xx - 1 = 6,7763$
$ly^2 = 6,8033868$	
$7,6343794$	
$lz^3 = 1,6555506$	
$5,9788288$	
$l32 = 1,5051500$	$\frac{(3xx-1)y^2}{32z^3} = 0,000002976$
$4,4736788$	

Ergo $f = 1,6117662$.

242. Methodus haec inveniendi radices aequationum proxime aeque patet ad quantitates tranſcendentes. Quaeramus numerum x, cuius logarithmus ex quocunque canone deſumptus ad ipſum numerum datam habeat rationem ut 1 ad n, atque habebitur iſta aequatio $x - nlx = 0$: ſit autem k modulus horum logarithmorum, ita ut iſti logarithmi obtineantur, ſi logarithmi hyperbolici multiplicentur per k, erit $d.lx = \frac{kdx}{x}$. Ponatur ergo $x - nlx = y$, ſitque f valor ipſius x quaeſitus, qui reddat $x = nlx$. Cum igitur ſit $y = x - nlx$, erit $dy = dx - \frac{kndx}{x} = \frac{dx(x-kn)}{x}$ &

$$\frac{dx}{dy} = p = \frac{x}{x-kn}; \text{ unde } dp = -\frac{kndx}{(x-kn)^2} \text{ ergo}$$

$$\frac{dp}{dy} = q = \frac{-knx}{(x-kn)^3}; \quad dq = \frac{2knxdx + k^2n^2dx}{(x-kn)^4}$$

$$\frac{dq}{dy} = r = \frac{knx(2x+kn)}{(x-kn)^5}; \text{ \&c.}$$ Quare fiet:

$$f = x - \frac{xy}{x-kn} - \frac{knxyy}{2(x-kn)^3} - \frac{knxy^3(2x+kn)}{6(x-kn)^5} - \text{\&c.}$$

Infra autem oſtendemus hoc problema ſolutionem non admittere, niſi ſit $kn > e$, exiſtente e numero cuius logarithmus hyperbolicus eſt $= 1$, ſeu debet eſſe $kn > 2,7182818$.

EXEMPLUM.

Quaeratur numerus praeter 10, *cuius logarithmus tabularis aequetur decimae parti ipſius numeri*:

Quia de logarithmis tabularibus quaeſtio inſtituitur, erit $k = 0,434294481903 25$, atque ob $n = 10$ habebitur $kn = 4,3429448190325$. Facto iam $x = 1$, erit $y = 1$, fietque $f = 1 + \frac{1}{3,3429} + \frac{2,1714724}{(3,3429)^3} - \&c.$ ſicque proxime erit $f = 1,37$. Statuatur ergo $x = 1,37$, erit $lx = 0,1367205671564 06$, & ob $y = x - 10lx$, erit $y = 0,0027943284350 3$. & $-x + kn = 2,9729448190325$. Fiat ergo

$$\begin{aligned} lx &= 0,1367205 \\ ly &= \underline{7,4462773} \\ & 7,5829978 \\ l(kn - x) &= \underline{0,4731866} \\ & 7,1098112 \quad \frac{-xy}{x - kn} = 0,00128769. \end{aligned}$$

Deinde cum ſit tertius terminus $\frac{knxyy}{2(x - kn)^3} = \frac{kny}{2(x - kn)^2} \cdot \frac{xy}{x - kn}$

erit:

$$\begin{aligned} l\frac{-xy}{x - kn} &= 7,1098112 \\ ly &= 7,4462773 \\ lkn &= \underline{0,6377842} \\ & 5,1938727 \\ l(kn - x)^2 &= \underline{0,9463732} \\ & 4,2474995 \\ l2 &= \underline{0,3010300} \\ l \text{ tert. term.} &= 3,9464695 \\ \text{I. term. } x &= 1,37 \end{aligned}$$

II.

II. term. $= 0,00128769$
III. term. $= 0,00000088$

$$f = 1,37128857$$
$$lf = 0,137128857.$$

243. Si aequatio fuerit exponentialis, ea ad logarithmicam reduci poterit; ita ſi quaeratur valor ipſius x, ut ſit $x^x = a$, erit $xlx = la$. Quare poſito $y = xlx - la$, fiet

$$dy = dxlx + dx, \quad \& \quad \frac{dx}{dy} = p = \frac{1}{1 + lx}. \quad \text{Tumque}$$

$$dp = \frac{-dx}{x(1 + lx)^2}; \quad \& \quad \frac{dp}{dy} = q = \frac{-1}{x(1 + lx)^3},$$

$$dq = \frac{dx}{xx(1 + lx)^3} + \frac{3dx}{xx(1 + lx)^4}, \quad \text{ideoque}$$

$$\frac{dq}{dy} = r = \frac{1}{xx(1 + lx)^4} + \frac{3}{xx(1 + lx)^5}; \quad \text{porro erit}$$

$$dr = \frac{-2dx}{x^3(1 + lx)^4} - \frac{10dx}{x^3(1 + lx)^5} - \frac{15dx}{x^3(1 + lx)^6}; \quad \text{ergo}$$

$$s = \frac{-2}{x^3(1 + lx)^5} - \frac{10}{x^3(1 + lx)^6} - \frac{15}{x^3(1 + lx)^7}, \quad \&$$

$$t = \frac{6}{x^4(1 + lx)^6} + \frac{40}{x^4(1 + lx)^7} + \frac{105}{x^4(1 + lx)^8} + \frac{105}{x^4(1 + lx)^9}$$

$$u = \frac{-24}{x^5(1 + lx)^7} - \frac{196}{x^5(1 + lx)^8} - \frac{700}{x^5(1 + lx)^9} - \frac{1260}{x^5(1 + lx)^{10}} - \frac{945}{x^5(1 + lx)^{11}}.$$

Hinc ergo ſi verus valor ipſius x ſit $= f$, ita ut ſit $f^f = a$; erit:

$$f = x - \frac{y}{1 + lx} - \frac{yy}{2x(1 + lx)^3} - \frac{y^3}{2xx(1 + lx)^5} - \frac{5y^4}{8x^3(1 + lx)^7} - \frac{7y^5}{8x^4(1 + lx)^9}$$
$$- \frac{y^3}{6x^2(1 + lx)^4} - \frac{5y^4}{12x^3(1 + lx)^6} - \frac{7y^5}{8x^4(1 + lx)^8}$$

 —

$$-\frac{y^4}{12x^3(1+lx)^5}-\frac{y^5}{3x^4(1+lx)^7}$$

$$\&c. \qquad -\frac{y^5}{20x^4(1+lx)^6}$$

Haec ergo expreſſio in infinitum continuata, quicunque valor pro x ſtatuatur, ſumto $y = x\,lx - la$ verum ipſius f dabit valorem. Sic ſi ponatur $x = 1$, erit $y = -la$, &

$$f = 1 + la - \frac{(la)^2}{2} + \frac{2(la)^3}{3} - \frac{9(la)^4}{8} + \frac{32(la)^5}{15} - \frac{625(la)^6}{144} - \&c.$$

ubi notandum eſt eſſe la logarithmum hyperbolicum ipſius a.

EXEMPLUM.

Quaeratur numerus f, *ut ſit* $f^f = 100$.

Cum ſit $a = 100$, & $y = xlx - la = xlx - l100$, quia patet eſſe $f > 3$ & < 4, ſtatuatur

$$x = \tfrac{7}{2};\ \text{eritque}\ lx = 1,25276296849$$
$$xlx = 4,38467038972$$
$$l100 = 4,60517018599$$
$$y = -0,22049979627$$
$$1 + lx = 2,25276296849$$

Hinc erit logarithmis vulgaribus adhibendis:

$$l - y = 9,3434083$$
$$l(1+lx) = 0,3527156$$
$$8,9906927;\quad \frac{-y}{1+lx} = 0,0978797$$
$$ly^2 = 8,6868166$$
$$3l(1+lx) = 1,0581468$$
$$= 7,6286698$$
$$l2x = l7 = 0,8450980$$
$$6,7835718 \quad \frac{y^2}{2x(1+lx)^3} = 0,0006075$$

Ergo proxime erit $f = 3,5972722$, ſequentibus vero inſuper terminis ſumtis erit $f = 3,5972852$.

244. Praeterea autem calculus differentialis inſignem habet

bet uſum in reſolutione aequationum, ſi quaepiam relatio, quae inter radices intercedit, fuerit cognita. Sit propoſita aequatio $y=0$, in qua ſit y functio quaecunque ipſius x. Si iam verbi gratia conſtet, duas huius aequationis radices inter ſe differre quantitate data a hae duae radices facile invenientur ſequenti modo. Denotet x harum duarum radicum minorem, erit maior $= x+a$, quare cum functio y evaneſcat, ſi x ſignificet unam ex radicibus aequationis $y=0$, evaneſcet quoque y, ſi loco x ponatur $x+a$. Quocirca erit:

$$0 = y + \frac{ady}{dx} + \frac{a^2 ddy}{2dx^2} + \frac{a^3 d^3 y}{6dx^3} + \&c.$$

Unde cum ſit $y=0$ erit quoque

$$0 = \frac{dy}{dx} + \frac{addy}{2dx^2} + \frac{a^2 d^3 y}{6dx^3} + \frac{a^3 d^4 y}{24dx^4} + \&c.$$

quae duae aequationes ſimul ſumtae per methodum eliminationis dabunt valorem illius radicis x, quam alia radix ſuperat quantitate a.

EXEMPLUM.

Sit propoſita haec aequatio

$$x^5 - 24x^3 + 49xx - 36 = 0,$$

quam undecunque conſtet, habere duas radices unitate differentes.

Poſito $y = x^5 - 24x^3 + 49xx - 36$ erit:

$$\frac{dy}{dx} = 5x^4 - 72x^2 + 98x$$

$$\frac{ddy}{2dx^2} = 10x^3 - 72x + 49$$

$$\frac{d^3y}{6dx^3} = 10x^2 - 24$$

$$\frac{d^4y}{24dx^4} = 5x$$

$$\frac{d^5y}{120dx^5} = 1$$

Iam ob $a=1$ erit:

A

$$\text{A} \ldots \; 5x^4 + 10x^3 - 62x^2 + 31x + 26 = 0. \quad \text{At est}$$
$$\text{B} \ldots \; x^5 - 24x^3 + 49xx - 36 = 0.$$

Multiplicetur ſuperior per x & inferior per 5, alteraque ab altera ſubtracta relinquet:

$$10x^4 + 53x^3 - 214x^2 + 26x + 180 = 0 \quad \text{ſeu}$$
$$\text{C} \ldots \; 5x^4 + 29x^3 - 107x^2 + 13x + 90 = 0$$

a qua prima A ſubtracta relinquet:

$$\text{D} \ldots \; 19x^3 - 45x^2 - 18x + 64 = 0$$
$$\text{D}.5x \ldots \; 95x^4 - 225x^3 - 90x^2 + 320x = 0$$
$$\text{A}.19 \ldots \; \underline{95x^4 + 190x^3 - 1178x^2 + 589x + 494 = 0}$$
$$\text{E} \ldots\ldots \; 415x^3 - 1088x^2 + 269x + 494 = 0$$
$$\text{D}.415 \ldots \; 7885x^3 - 18675x^2 - 7470x + 26560 = 0$$
$$\text{E}.19 \ldots \; \underline{7885x^3 - 20672x^2 + 5111x + 9386 = 0}$$
$$\text{F} \ldots\ldots \; 1997x^2 - 12581x + 17174 = 0$$
$$\text{D}.247 \ldots \; 4693x^3 - 11115x^2 - 4446x + 15808 = 0$$
$$\text{E}.32 \ldots \; \underline{13280x^3 - 34816x^2 + 8608x + 15808 = 0}$$
$$\underline{8587x^3 - 23701x^2 + 13054x = 0}$$
$$\text{G} \ldots\ldots \; 8587x^2 - 23701x + 13054 = 0$$
$$\text{F}.8587 \ldots \; 17148239x^2 - 108033047x + 147473138 = 0$$
$$\text{G}.1997 \ldots \; \underline{17148239x^2 - 47330897x + 26068838 = 0}$$
$$60702150x - 121404300 = 0$$

Ex qua aequatione ſequitur $x = 2$,
ac propterea quoque radix aequationis erit $x = 3$, quorum uterque valor aequationis ſatisfacit.

245. Poteſt autem haec operatio abſolvi ſine ſubſidio calculi differentialis, propterea quod eadem aequatio, quam calculus differentialis ſuppeditavit, prodit ſi in ipſa aequatione propoſita ponatur $x + a$ loco x. Ceterum vero haec methodus eliminandi nimium eſt operoſa, &, ſi aequationes eſſent altioris gradus, labor penitus foret inſuperabilis; ex quo multo minus in aequationibus tranſcendentibus locum habere poteſt. Quod ſi autem ponamus duas aequationis propoſitae $y = 0$ radices inter ſe eſſe aequales, tum ob $a = 0$, aequa-

aequatio differentialis abit in hanc $\frac{dy}{dx} = 0$. Quoties ergo quaepiam aequatio $y = 0$ habuerit duas radices aequales, toties erit $\frac{dy}{dx} = 0$; atque hae duae aequationes coniunctae praebebunt eum ipſius x valorem, cui binae radices ſunt aequales. Unde viciſſim ſi ambae aequationes $y = 0$ & $\frac{dy}{dx} = 0$ communem habeant radicem, ea erit radix duplex aequationis $y = 0$. Evenit autem hoc, ſi poſtquam quantitas x ope duarum iſtarum aequationum $y = 0$ & $\frac{dy}{dx} = 0$ penitus fuerit eliminata, perveniatur ad aequationem identicam. Sic ſi proponatur aequatio:

$$x^3 - 2xx - 4x + 8 = 0$$

erit quoque $3xx - 4x - 4 = 0$, cuius duplum ad eam additum dat $x^3 + 4xx - 12x = 0$ ſeu $xx + 4x - 12 = 0$ cuius triplum eſt $3xx + 12x - 36 = 0$

ſubtrahatur $3xx - 4x - 4 = 0$

$$16x - 32 = 0$$

$$x - 2 = 0$$

Cum ergo prodierit $x = 2$, ſubſtituatur hic valor in una praecedentium $3xx - 4x - 4 = 0$, & prodibit aequatio identica $12 - 8 - 4 = 0$, unde colligitur aequationem propoſitam $x^3 - 2xx - 4x + 8 = 0$ duas habere radices aequales; nempe 2.

246. Si igitur habeatur aequatio algebraica quotcumque dimenſionum:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \&c. = 0$$

quae duas habeat radices inter ſe aequales, erit quoque

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + (n-4)Dx^{n-5} + \&c. = 0.$$

Scilicet illius aequationis radix duplex ſimul erit radix iſtius aequationis. Multiplicetur illa per n, ab eaque haec per x multiplicata ſubtrahatur, prodibitque haec nova aequatio:

$Ax^{n-1} + 2Bx^{n-2} + 3Cx^{n-3} + 4Dx^{n-4} + \&c. = 0$. Nunc addantur prima per a & haec per b multiplicatae erit: $ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \&c. = 0$ quae aequatio cum ipſa propoſita coniuncta monſtrabit radices aequales, ſi quas habet propoſita. Cum igitur quantitates a & b pro lubitu aſſumi queant, coefficientes a, $a+b$, $a+2b$, &c. progreſſionem quamcunque arithmeticam repraeſentant. Quamobrem ſi aequatio quaecunque habeat duas radices aequales, eae invenientur, ſi ſinguli aequationis propoſitae termini multiplicentur per terminos cuiuſvis progreſſionis arithmeticae reſpective; nova enim aequatio hoc modo reſultans eam radicem, quae in propoſita bis ineſt, quoque continebit. Sic aequatio:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \&c. = 0$$

ſi eius termini multiplicentur per progreſſionem arithmeticam hanc: a; $a+b$; $a+2b$; $a+3b$; $a+4b$; &c. prodibit nova aequatio haec:

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \&c. = 0$$

quae cum illa coniuncta radices aequales oſtendet. Haecque eſt regula ſatis cognita inveniendi radices aequales cuiuſcunque aequationis.

247. Si aequatio $y = 0$ tres habeat radices aequales non ſolum erit $\frac{dy}{dx} = 0$, ſed etiam erit $\frac{ddy}{dx^2} = 0$; ſi quidem pro x ſtatuatur eius radicis valor, quae in aequatione $y = 0$ ter ineſt. Ad hoc oſtendendum ponamus aequationem $y = 0$ tres habere radices huiuſmodi x, $x+a$, & $x+b$, quae primum intervallis finitis a & b a ſe invicem diſcrepent; & quia y evaneſcit, ſi loco x tam $x+a$, quam $x+b$ ſcribatur, erit: $y = 0$

$$y + \frac{ady}{dx} + \frac{a^2ddy}{2dx^2} + \frac{a^3d^3y}{6dx^3} + \frac{a^4d^4y}{24dx^4} + \&c. = 0$$

$$y + \frac{bdy}{dx} + \frac{b^2ddy}{2dx^2} + \frac{b^3d^3y}{6dx^3} + \frac{b^4d^4y}{24dx^4} + \&c. = 0$$

a qui-

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a quibus binis poſterioribus ſi prima ſubtrahatur erit:

$$\frac{dy}{dx}+\frac{addy}{2dx^2}+\frac{a^2d^3y}{6dx^3}+\frac{a^3d^4y}{24dx^4}+\&c.=0$$

$$\frac{dy}{dx}+\frac{bddy}{2dx^2}+\frac{b^2d^3y}{6dx^3}+\frac{b^3d^4y}{24dx^4}+\&c.=0$$

Subtrahantur quoque hae a ſe invicem, diviſioneque per $a-b$ facta erit:

$$\frac{ddy}{2dx^2}+\frac{(a+b)d^3y}{6dx^3}+\frac{(aa+ab+bb)d^4y}{24dx^4}+\&c.=0.$$

Ponatur iam $a=0$ & $b=0$, ita ut tres illae radices inter ſe ſint aequales, eritque ob terminos evaneſcentes:

$$y=0;\ \frac{dy}{dx}=0;\ \&\ \frac{ddy}{dx^2}=0.$$

248. Quoties ergo aequatio $y=0$, tres habeat radices aequales puta f, f, f, tum iſta quantitas f erit quoque radix non ſolum huius aequationis $\frac{dy}{dx}=0$, ſed etiam huius $\frac{ddy}{dx^2}=0$. Hinc manifeſtum eſt, cum f ſit radix communis aequationis $\frac{dy}{dx}=0$, & eius differentialis $\frac{ddy}{dx^2}=0$, eam in aequatione $\frac{dy}{dx}=0$, bis ineſſe debere, per ea quae ante de binis radicibus aequalibus oſtendimus. Quare ſi aequatio:

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+Dx^{n-4}+\&c.=0$$

tres contineat radices aequales f, f, f, ſi eius termini per terminos progreſſionis arithmeticae cuiuſvis multiplicentur, tum aequatio reſultans binas habebit radices aequales f & f: quamobrem ea denuo per progreſſionem arithmeticam quamcunque multiplicari poterit, ut prodeat aequatio eamdem radicem f ſemel complectens. Obtinebuntur ergo tres aequationes communem radicem f habentes, ex quarum combina-

tione haec ipſa radix facile elicietur. Si enim eiuſmodi progreſſiones arithmeticae eligantur, quarum vel primus vel ultimus terminus ſit $=0$, tum aequatio prodibit uno gradu inferior, ſicque eliminatio eo facilior evadet.

249. Simili modo oſtendetur, ſi aequatio $y=0$ quatuor habeat radices aequales f, f, f, f, tum poſito $x=f$ non ſolum fieri $y=0$, $\frac{dy}{dx}=0$, & $\frac{ddy}{dx^2}=0$, ſed etiam fore $\frac{d^3y}{dx^3}=0$. Scilicet uti aequatio $y=0$ quater continet radicem $x=f$; ita aequatio $\frac{dy}{dx}$ eandem radicem ter; aequatio vero $\frac{ddy}{dx^2}=0$ bis, & aequatio $\frac{d^3y}{dx^3}=0$ ſemel complectetur. Hoc quoque facilius perſpicietur, ſi perpendamus functionem y hoc caſu huiuſmodi formam $(x-f)^4 X$ habere debere, denotante X functionem quamcumque ipſius x. Hac forma aſſumta erit: $\frac{dy}{dx}=(x-f)^3\left(4X+\frac{(x-f)dX}{dx}\right)$, ideoque per $(x-f)^3$ diviſibilis. Similiter porro habebit $\frac{ddy}{dx^2}$ factorum $(x-f)^2$, & $\frac{d^3y}{dx^3}$ factorem $x-f$; ex quo perſpicuum eſt, ſi radix $x=f$ in aequatione $y=0$ quater inſit, eam in aequatione $\frac{dy}{dx}=0$ ter, in aequatione $\frac{ddy}{dx^2}=0$ bis, atque in $\frac{d^3y}{dx^3}=0$ ſemel adhuc ineſſe debere.

CA-

CAPUT X.

DE MAXIMIS ET MINIMIS.

250.

Si functio ipſius x ita fuerit comparata, ut creſcentibus valoribus x ipſa continuo creſcat vel decreſcat, tum iſta functio nullum habebit valorem maximum minimumve. Quicunque enim huius functionis valor conſideretur, ſequentes erunt maiores, praecedentes vero minores. Huiuſmodi functio eſt $x^3 + x$, cuius valor creſcentibus x continuo creſcit, decreſcentibus vero x, continuo decreſcit; maximum ergo haec functio valorem alium induere nequit, niſi ipſi x valor maximus, hoc eſt infinitus tribuatur; ſimilique modo minimum obtinebit valorem, ſi ponatur $x = -\infty$. Niſi autem functio ita fuerit comparata, ut creſcente x continuo creſcat decreſcatve, maximum vel minimum alicubi habebit valorem; hoc eſt eiuſmodi valorem, qui ſit vel maior vel minor, quam antecedentes & ſequentes. Sic iſta functio $xx - 2x + 3$ minimum valorem induit, ſi ponatur $x = 1$, quicunque enim alius valor ipſi x tribuatur, perpetuo functio maiorem adipiſcetur valorem.

251. Quo autem natura maximorum ac minimorum clarius perſpiciatur, ponamus y eiuſmodi eſſe functionem ipſius x, quae maximum obtineat valorem, ſi ponatur $x = f$; atque intelligitur, ſi x ponatur ſive maius ſive minus quam f, tum valorem ipſius y inde oriundum minorem fore illo, quem induit ſi ponatur $x = f$. Simili modo, ſi poſito $x = f$, functio y minimum obtineat valorem, neceſſe eſt ut, ſive x ponatur maius quam f ſive minus, ſemper maior ipſius y valor reſultet: haecque eſt definitio maximorum & minimorum abſolutorum. Praeterea autem quoque functio y maxi-

mum valorem recipere dicitur poſito verbi gratia $x = f$, dummodo iſte valor maior fuerit quam proximi ſive ſequentes ſive antecedentes, qui oriuntur, ſi x aliquantillum ſive maius ſive minus quam f ſtatuatur; etiamſi aliis valoribus loco x ſubſtituendis functio y maiores forte valores recipiat. Similiter functio y minimum valorem recipere dicitur poſito $x = f$, dummodo ille valor minor fuerit iis, quos induit, ſi loco x valores proxime ſive maiores ſive minores quam f ſubſtituantur. Atque in hac poſteriori ſignificatione iſtis maximorum & minimorum vocabulis utemur.

252. Antequam autem modum oſtendamus haec maxima & minima inveniendi, notari convenit hanc inveſtigationem proprie in iis tantum ipſius x functionibus locum habere, quas ſupra uniformes vocavimus, & quae ſunt ita comparatae, ut pro ſingulis ipſius x valoribus ſingulos pariter valores recipiant. Biformes autem & multiformes functiones vocavimus, quae pro ſingulis valoribus ipſius x binos pluresve valores inducunt, cuiuſmodi functiones ſunt radices aequationum quadraticarum & plurium dimenſionum. Si igitur y huiuſmodi fuerit functio ipſius x vel biformis vel multiformis, tum proprie dici nequit, eam poſito $x = f$ valorem ſive maximum ſive minimum induere: quoniam enim poſito $x = f$ vel duos pluresve valores ſimul obtinet, atque praecedentes aeque ac ſequentes ſint numero plures, diiudicatio maximi minimive non tam facile inſtituitur: niſi forte omnes functionis y valores, qui ſingulis ipſius x valoribus reſpondent, ſint imaginarii praeter unum; quo caſu huiuſmodi functiones ſpeciem functionum uniformium mentiuntur. Primum ergo functiones uniformes harumque ſpeciem mentientes contemplabimur, tum vero quomodo iudicium ad multiformes accommodari debeat, indicabimus.

253. Sit igitur y functio ipſius x uniformis, quae propterea, quicunque valor pro x ſubſtituatur, ſemper unum recipiat valorem realem, denotetque x eum valorem, qui fun-

functioni y maximum minimumve valorem inducat. Priori ergo caſu, ſive loco x ſubſtituatur $x+\alpha$ ſive $x-\alpha$, valor ipſius y minor erit, quam ſi $\alpha=0$, poſteriori vero caſu maior. Cum igitur poſito $x+\alpha$ loco x functio y abeat in

$$y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3y}{6dx^3}+\&c.$$

at poſito $x-\alpha$ loco x in

$$y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3y}{6dx^3}+\&c.$$

neceſſe eſt ut caſu maximi ſit

$$y>y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3y}{6dx^3}+\&c. \quad \&$$

$$y>y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3y}{6dx^3}+\&c.$$

In caſu autem, quo valor ipſius y ſit minimus erit:

$$y<y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3y}{6dx^3}+\&c.$$

$$y<y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3y}{6dx^3}+\&c.$$

254. Quoniam haec evenire debent, ſi α denotet quantitatem minimam, ſtatuamus α tam parvum, ut eius poteſtates altiores reiici queant; debebitque tam pro caſu maximi quam minimi eſſe $\frac{\alpha dy}{dx}=0$. Niſi enim $\frac{\alpha dy}{dx}$ eſſet $=0$, neque valor ipſius y maximus neque minimus eſſe poſſet. Hinc tam pro maximis quam pro minimis inveſtigandis haec habetur regula communis, ut differentiale propoſitae y nihilo aequale ponatur, eritque ille ipſius x valor, qui functionem reddit vel maximam vel minimam, radix iſtius aequationis. Utrum vero valor hoc modo inventus ipſius y futurus ſit maximus an minimus, incertum relinquitur; quin etiam fieri poteſt, ut y neque maximum neque minimum

ſit

ſit futurum: tantum enim invenimus utroque caſu fore $\frac{dy}{dx}=0$, neque viciſſim affirmavimus, quoties ſit $\frac{dy}{dx}=0$, toties quoque valorem pro y prodire vel maximum vel minimum.

255. Interim tamen ad caſus, quibus valor ipſius vel maximus vel minimus evadat, inveſtigandos haec prima operatio inſtituenda eſt, ut differentiale functionis propoſitae nihilo aequetur, atque ex aequatione $\frac{dy}{dx}=0$ omnes ipſius x valores eliciantur. Quibus inventis deinceps diſpiciendum erit, utrum iis functio y maximum induat valorem, an minimum, an neutrum; oſtendemus enim fieri poſſe, ut neque maximum neque minimum locum habeat, etiamſi ſit $\frac{dy}{dx}=0$. Sit f valor ſeu unus ex valoribus ipſius x, quem obtinet ex aequatione $\frac{dy}{dx}=0$, hicque valor ſubſtituatur in expreſſionibus $\frac{ddy}{dx^2}$; $\frac{d^3y}{dx^3}$; &c. fiatque hac ſubſtitutione $\frac{ddy}{dx^2}=p$; $\frac{d^3y}{dx^3}=q$; $\frac{d^4y}{dx^4}=r$ &c. Abeat autem functio ipſa y poſito f loco x in F, atque ſi loco x ponatur $f+a$, iſta functio abibit in

$$F+\frac{1}{2}a^2p+\frac{1}{6}a^3q+\frac{1}{24}a^4r+\&c.$$

ſin autem loco x ponatur $f-a$ prodibit

$$F+\frac{1}{2}a^2p-\frac{1}{6}a^3q+\frac{1}{24}a^4r-\&c.$$

unde patet, ſi p fuerit quantitas affirmativa, utrumque valorem maiorem fore quam F, ſaltem ſi a quantitatem valde parvam

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parvam denotet, ac propterea valorem F, quem functio y induit poſito $x = f$, fore minimum. Sin autem p ſit quantitas negativa, tum valor $x = f$ functioni y inducet valorem maximum.

256. Quodſi autem fuerit $p = 0$, tum ſpectari debet valor ipſius q, qui ſi non fuerit $= 0$, valor ipſius y neque maximus erit neque minimus; nam poſito $x = f + \alpha$ erit $F + \frac{1}{6}\alpha^3 q > F$ & poſito $x = f - \alpha$ erit $F - \frac{1}{6}\alpha^3 q < F$. Sin autem quoque fuerit $q = 0$, ad quantitatem r erit reſpiciendum, quae ſi habuerit valorem affirmativum valor functionis F, quem recipit poſito $x = f$, erit minimum; ſin autem r habeat valorem negativum, erit F maximum. At ſi quoque r evaneſcat iudicium ex ſequentis litterae s valore erit petendum, quod ſimile erit illi, quod ex littera q formavimus. Scilicet ſi s non fuerit $= 0$, functio F neque maximum erit neque minimum; ſin autem ſit quoque $s = 0$, tum ſequens littera t, ſi habeat valorem affirmativum indicabit minimum, ſin autem habeat valorem negativum, indicabit maximum. Verum ſi & haec littera t evaneſcat, tum in iudicando ulterius eſt procedendum eodem prorſus modo, quo in caſibus praecedentibus ſumus uſi. Sicque de qualibet radice aequationis $\frac{dy}{dx} = 0$ indagabitur, utrum functioni y inducat valorem maximum an minimum, an neutrum; atque hoc modo omnia maxima & minima, quae quidem functio y recipere poteſt, invenientur.

257. Si ergo aequatio $\frac{dy}{dx} = 0$ duas radices habeat aequales, ita ut factorem habeat quadratum $(x - f)^2$, tum poſito $x = f$ ſimul $\frac{ddy}{dx^2}$ evaneſcet, eritque $p = 0$, non au-

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tem q. Hoc ergo caſu functio y neque maximum neque minimum valorem induet. Sin autem aequatio $\frac{dy}{dx} = 0$ tres radices habeat aequales, ſeu $\frac{dy}{dx}$ factorem cubicum $(x - f)^3$ r tum poſito $x = f$, fiet $\frac{ddy}{dx^2} = 0$ & $\frac{d^3y}{dx^3} = 0$; non autem $\frac{d^4y}{dx^4}$. Huius ergo termini valor ſi fuerit affirmativus indicabit minimum, ſin negativus, maximum. Iudicium ergo ante explicatum huc redit, ut ſi expreſſio $\frac{dy}{dx}$ factorem habuerit $(x-f)^n$, exiſtente n numero impari, functio y, ſi in ea ponatur $x = f$, valorem ſit acceptura vel maximum vel minimum; ſin autem exponens n fuerit numerus par, tum ſubſtitutio $x = f$ neque maximum neque minimum valorem producat.

258. Deinde inventio maximi ac minimi ſaepenumero non mediocriter adiuvabitur ſequentibus conſiderationibus. Quibus ſcilicet caſibus functio y fit maximum vel minimum, iiſdem caſibus fiet quodvis eius multiplum ay, ſi quidem a fuerit quantitas affirmativa, itemque y^3, y^5, y^7, &c. atque generaliter ay^n, ſi quidem n fuerit numerus affirmativus impar, pariter maximum vel minimum; quoniam huiuſmodi formulae ita ſunt comparatae, ut creſcente y creſcant, & decreſcente y, decreſcant. Quibus autem caſibus fit y maximum vel minimum, iiſdem caſibus $-y$, $-ay$, $b - ay$, & generaliter $b - ay^n$ exiſtente n numero affirmativo impari, fiet ordine inverſo vel minimum vel maximum. Similiter quibus caſibus y fit maximum vel minimum, iiſdem caſibus formulae hae $\frac{a}{y}$, $\frac{a}{y^3}$, $\frac{a}{y^5}$, & generaliter $\frac{a}{y^n} \pm b$, denotante a quantitatem affirmativam & n numerum affirmativum imparem, fient inverſo ordine vel minimum vel maximum; ſin autem a fuerit quantitas negativa, tum iſtae formulae maximum impetrabunt valorem, ſi y fuerit maximum, & minimum, ſi y fit minimum.

259.

259. Ad poteſtates autem pares haec non item traduci poſſunt: quoniam enim, ſi y valores recipit negativos, eius poteſtates pares y^2, y^4, &c. valores affirmativos inducunt, fieri poteſt, ut dum y minimum valorem negativum ſcilicet recipit, eius poteſtates pares fiant maxima. Huius igitur conditionis ratione habita affirmare poterimus, ſi y fuerit maximum vel minimum, exiſtente eius valore affirmativo, tum eius poteſtates pares y^2, y^4 &c. quoque fore maxima, vel minima. Sin autem valor ipſius y negativus fuerit maximum, tum eius quadratum yy accepturum eſſe valorem minimum, & contra ſi valor ipſius y negativus ſit minimum, tum y^2, y^4, &c. fore maximum. Quodſi vero exponentes ipſius y pares fuerint negativi, tum contrarium eveniet. Ceterum quae hic de exponentibus paribus & imparibus annotavimus, ea non ſolum pro numeris integris valent, ſed etiam pro fractis, quorum denominatores ſunt numeri impares; in hoc enim negotio fractiones $\frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{1}{5}, \frac{3}{5}$, &c. numeris imparibus, at $\frac{2}{3}, \frac{4}{3}, \frac{2}{5}, \frac{4}{5}, \frac{6}{7}$, &c. numeris paribus aequivalent.

260. Sin autem denominatores fuerint numeri pares, tum, quoniam ſi y negativum habet valorem, eius poteſtates $y^{\frac{1}{2}}$; $y^{\frac{3}{4}}$; &c. fiunt imaginariae; hoc tantum de iis affirmari poterit, ſi valor ipſius y affirmativus fuerit maximum vel minimum, tum quoque $y^{\frac{1}{2}}$; $y^{\frac{3}{2}}$; $y^{\frac{1}{4}}$; &c. fore pariter vel maxima vel minima; contra autem $y^{-\frac{1}{2}}$, $y^{-\frac{3}{2}}$, $y^{-\frac{1}{4}}$, &c. minima vel maxima. Quia autem haec irrationalia ſimul geminos valores habent, alterum affirmativum, alterum negativum, de negativis contrarium erit tenendum, quod hic de affirmativis diximus. Sin autem valor ipſius y negativus

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evadat maximum vel minimum, tum quia huiuſmodi poteſtates omnes fiunt imaginariae, neque maximis neque minimis annumerari poterunt. His igitur ſubſidiis inveſtigatio maximi & minimi ſaepe admodum reddetur facilis, quae alias futura eſſet vehementer difficilis.

261. Quoniam haec proprie ad functiones rationales, quippe quae ſunt ſolae uniformes, pertinent, primum functiones integras evolvamus, atque maxima minimaque quae in ipſis occurrunt indagemus. Cum igitur huiuſmodi functiones ad hanc formam referantur:

$$x^n + A x^{n-1} + B x^{n-2} + C x^{n-3} + \&c.$$

primum patet earum valorem maiorem fieri non poſſe, quam ſi ponatur $x = \infty$; tum vero ſi $x = -\infty$ valor huius formulae prodit $= \infty^n$, ſi n ſit numerus par, at $-\infty^n$ ſi n ſit numerus impar, qui propterea valor erit omnium minimus. Dantur autem praeterea ſaepe alia maxima & minima eo ſenſu, quem his vocibus attribuimus, quae ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Invenire valores ipſius x, *quibus haec functio* $(x-a)^n$ *ſit maximum vel minimum.*

Poſito $(x-a)^n = y$, erit $\frac{dy}{dx} = n(x-a)^{n-1}$, quo poſito $= 0$, fiet $x = a$. Cum igitur $\frac{dy}{dx}$ factorem habeat $(x-a)^{n-1}$, ex §. 257. intelligitur y maximum minimumve eſſe non poſſe, niſi ſit $n-1$ numerus impar, ſeu n numerus par. Quia autem dum ſit

$$\frac{d^n y}{dx^n} = n(n-1)(n-2) \ldots 1$$

hoc eſt numerus affirmativus, ſequitur, valorem ipſius y poſito $x = a$ proditurum eſſe minimum. Quod quidem facile patet; nam poſito $x = a$ fit $y = 0$; & ſi x ponatur vel maius

maius vel minus quam a, ob n numerum parem, accipiet y valorem poſitivum hoc eſt nihilo maiorem; ſin autem n fuerit numerus impar, tum functio $y = (x-a)^n$ neque maximum neque minimum admittit. Perſpicuum autem porro eſt hoc idem valere, ſi n fuerit numerus fractus ſive impar ſive par. Scilicet $(x-a)^{\frac{\mu}{\nu}}$ fiet poſito $x = a$ minimum, ſi μ fuerit numerus par & ν impar; ſin autem uterque fuerit impar, neque maximum dabitur neque minimum.

EXEMPLUM II.

Invenire caſus quibus valor huius formulae $xx + 3x + 2$ *ſit maximum vel minimum.*

Ponatur $xx + 3x + 2 = y$, erit $\frac{dy}{dx} = 2x + 3$, $\frac{ddy}{2dx^2} = 1$. Statuatur ergo $2x + 3 = 0$, fiet $x = -\frac{3}{2}$; qui caſus utrum maximum an minimum producat, cognoſcetur ex valore $\frac{ddy}{2dx^2} = 1$, qui cum ſit affirmativus, quicquid ſit x, indicat minimum. Poſito autem $x = -\frac{3}{2}$ fit $y = -\frac{1}{4}$, & ſi alii quicumque valores ipſi x tribuantur, valor ipſius y inde oriundus perpetuo maior erit quam $-\frac{1}{4}$. Ex natura quoque ipſius formulae $xx + 3x + 2$ perſpicitur, eam minimum valorem habere debere; nam cum in infinitum excreſcat ſive ponatur $x = +\infty$ ſive $x = -\infty$, neceſſe eſt, ut quiſpiam valor ipſius x ipſi y omnium minimam quantitatem inducat.

EXEMPLUM III.

Invenire caſus, quibus expreſſio haec $x^3 - axx + bx - c$ *maximum minimumve valorem accipit.*

Poſito $y = x^3 - axx + bx - c$; erit $\frac{dy}{dx} = 3xx - 2ax + b$ & $\frac{ddy}{2dx^2} = 3x - a$; $\frac{d^3y}{6dx^3} = 1$. Statuatur ergo

$\frac{dy}{dx} = 3xx - 2ax + b = 0$; erit $x = \frac{a \pm \surd(aa - 3b)}{3}$, ex quo intelligitur, niſi ſit $aa > 3b$, formulam propoſitam neque maximum neque minimum eſſe habituram. Sin autem ſit $aa > 3b$, duobus caſibus fit maximum vel minimum. Hinc vero oritur $\frac{ddy}{2dx^2} = \pm \surd(aa - 3b)$, unde intelligitur niſi ſit $aa = 3b$, valorem $x = \frac{a + \surd(aa - 3b)}{3}$ reddere formulam $y = x^3 - axx + bx - c$ minimam; alterum vero $x = \frac{a - \surd(aa - 3b)}{3}$ maximam. Quanti autem futuri iſti ſint ipſius y valores, cum ſit $3xx - 2ax + b = 0$ ſeu $x^3 - \frac{2}{3}axx + \frac{1}{3}bx = 0$; erit $y = -\frac{1}{3}axx + \frac{2}{3}bx - c$ & ob $\frac{1}{3}axx - \frac{2aa}{9}x + \frac{ab}{9} = 0$, fit $y = \frac{2}{9}(3b - aa)x + \frac{ab}{9} - c = -\frac{2a(aa - 3b)}{27} \mp \frac{2(aa - 3b)\surd(aa - 3b)}{27} + \frac{ab}{9} - c$, ſive $y = -\frac{2a^3}{27} + \frac{ab}{3} - c \mp \frac{2}{27}(aa - 3b)^{\frac{3}{2}}$, ubi ſignum ſuperius valet pro minimo, inferius autem pro maximo. Reſtat ergo caſus quo $aa = 3b$, in quo cum fiat $\frac{ddy}{2dx^2} = 0$, ſequens vero terminus $\frac{d^3y}{6dx^3} = 1$, non ſit $= 0$, ſequitur hoc caſu formulam propoſitam neque maximum neque minimum eſſe recepturam.

EXEMPLUM IV.

Invenire caſus quibus haec functio ipſius x;
$x^4 - 8x^3 + 22x^2 - 24x + 12$
fit maximum vel minimum.

Poſito $y = x^4 - 8x^3 + 22x^2 - 24x + 12$; erit

dy

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24$$

$$\frac{ddy}{2dx^2} = 6x^2 - 24x + 22$$

Statuatur nunc $\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0$ ſeu $x^3 - 6x^2 + 11x - 6 = 0$, orientur tres valores reales pro x; I. $x = 1$; II. $x = 2$; III. $x = 3$. Ex primo valore fit $\frac{ddy}{2dx^2} = 4$, ideoque poſito $x = 1$ functio propoſita fit minimum. Ex ſecundo valore $x = 2$ fit $\frac{ddy}{2dx^2} = -2$, ideoque functio propoſita maximum. Ex tertio valore $x = 3$ fit $\frac{ddy}{2dx^2} = +4$, ideoque functio propoſita iterum minimum.

EXEMPLUM V.

Propoſita ſit haec functio $y = x^5 - 5x^4 + 5x^3 + 1$, *quae, quibus caſibus fiat maximum minimumve quaeritur.*

Cum ſit $\frac{dy}{dx} = 5x^4 - 20x^3 + 15xx$, formetur aequatio $x^4 - 4x^3 + 3xx = 0$, cuius radices ſunt I. & II. $x = 0$; III. $x = 1$; IV. $x = 3$. Quoniam prima & ſecunda radices ſunt aequales, ex iis neque maximum neque minimum ſequitur, fit enim $\frac{ddy}{dx^2} = 0$, at $\frac{d^3y}{2dx^3}$ non evaneſcit. Tertia radix autem $x = 1$, ob $\frac{ddy}{2dx^2} = 10x^3 - 30x^2 + 15x$ praebet $\frac{ddy}{2dx^2} = -5$, hocque ergo caſu functio fit maximum. Ex quarta radice $x = 3$ fit $\frac{ddy}{2dx^3} = 45$, ideoque functio propoſita minimum.

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EXEMPLUM VI.

Invenire caſus quibus haec formula

$$y = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$$

ſit maximum vel minimum.

Erit ergo $\frac{dy}{dx} = 60x^5 - 60x^4 + 60x^3 - 60x^2$, & $\frac{ddy}{60dx^2} = 5x^4 - 4x^3 + 3x^2 - 2x$. Formetur aequatio $x^5 - x^4 + x^3 - xx = 0$, quae cum in factores reſoluta ſit $x^2(x-1)(xx+1) = 0$, duas habet radices aequales $x = 0$, & praeterea radicem $x = 1$, duaſque inſuper ex $xx + 1 = 0$ imaginarias. Cum igitur binae radices aequales $x = 0$ neque maximum neque minimum exhibeant, tantum conſideranda ſupereſt radix $x = 1$, ex qua fit $\frac{ddy}{60dx^2} = 2$, cuius valor affirmativus indicat minimum.

262. Determinatio ergo maximorum & minimorum pendet a radicibus aequationis differentialis $\frac{dy}{dx} = 0$, cuius poteſtas ſumma, cum ſit uno gradu inferior quam in ipſa functione propoſita y, ſi quidem haec fuerit functio rationalis integra: manifeſtum eſt ſi in genere proponatur haec functio:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \&c. = y.$$

eius maxima & minima determinari per radices huius aequationis:

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + \&c. = 0.$$

Ponamus huius aequationis radices reales ſecundum ordinem quantitatis diſpoſitas eſſe α, β, γ, δ, &c. ita ut α ſit maxima $\beta < \alpha$, $\gamma < \beta$, &c. Ac primo quidem ſi hae radices omnes fuerint inaequales, unaquaeque formulae propoſitae y inducet valorem maximum vel minimum; totque idcirco functio y habebit maxima vel minima, quo aequatio $\frac{dy}{dx} = 0$ ha-

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habuerit radices reales inaequales. Sin autem duae pluresve radices inter ſe fuerint aequales, res ita ſe habebit, ut duae radices aequales neque maximum neque minimum exhibeant; ternae vero radices aequales uni aequivaleant: atque in genere ſi numerus radicum aequalium fuerit par, nullum inde reſultat maximum minimumve; ſin autem ſit impar, unum inde oritur ſive maximum ſive minimum.

263. Quaenam autem radices maxima, & quae minima producant, ſine ſubſidio regulae ante traditae ita definiri poterit. Cum functio y poſito $x = \infty$ fiat pariter infinita, neque valores ipſius x intra limites ∞ & α, ullum producant ſive maximum ſive minimum; perſpicuum eſt valores functionis y, dum loco x ſucceſſive valores ab ∞ uſque ad α ſubſtituantur, continuo decreſcere oportere; ideoque valor $x = \alpha + \omega$ functioni y maiorem valorem inducet, quam valor $x = \alpha$: unde cum $x = \alpha$ maximum minimumve producat, neceſſe eſt, ut hoc caſu functio y fiat minimum. Ulterius ergo diminuendo ſeu ponendo $x = \alpha - \omega$, valor ipſius y iterum creſcet, donec fiat $x = \beta$, quae eſt ſecunda aequationis $\frac{dy}{dx} = 0$ radix maximum minimumve producens: quare haec ſecunda radix $x = \beta$ maximum praebebit, & valor $x = \beta - \omega$ minorem efficiet functionem y, quam $x = \beta$, donec perveniatur ad $x = \gamma$, quae conſequenter iterum minimum generabit. Ex quo ratiocinio perſpicitur, radices aequationis $\frac{dy}{dx} = 0$ primam, tertiam quintam, &c. minima, ſecundam autem, quartam, ſextam &c. maxima exhibere. Simul autem hinc intelligitur in caſu duarum radicum aequalium maximum & minimum coaleſcere, ſicque neutrum locum habere.

264. Si ergo in functione propoſita

$$y = x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c.$$

maximus exponens n fuerit numerus par, aequatio

$$\frac{dy}{dx} = x^{n-1} + (n-1)Ax^{n-2} + \&c. = 0;$$

erit gradus imparis, ideoque vel unam habebit radicem realem, vel tres, vel quinque, vel numero impares. Si unica radix fuerit realis, ea dabit minimum; ſin tres fuerint reales, maxima praebebit minimum, media maximum, & minima iterum minimum; & ſi quinque radices fuerint reales, functio y tria habebit minima & duo maxima; ſicque porro. At ſi exponens n fuerit numerus impar, aequatio $\frac{dy}{dx} = 0$ ad gradum parem pertinebit, ideoque vel nullam habebit radicem realem, vel duas, vel quatuor, vel ſex, &c. Primo caſu functio y neque maximum habebit neque minimum: altero caſu, quo duae dantur radices, earum maior minimum, minor autem maximum indicabit: quatuor autem radicum prima (quae eſt maxima) & tertia minimum, ſecunda vero & quarta maximum producunt. Perpetuo autem quotcunque radices fuerint reales, maxima & minima ſe mutuo alternatim inſequuntur.

265. Progrediamur ad functiones rationales fractas, quibus altera ſpecies functionum uniformium conſtituitur. Sit igitur $y = \frac{P}{Q}$, exiſtentibus P & Q functionibus quibuſcunque ipſius x; ac primo quidem apparet, ſi ipſi x eiuſmodi valor tribuatur, ut fiat $Q = 0$, niſi ſimul P evaneſcat, functionem y evadere infinitam, quod utique maximum videatur. Nihilo vero minus iſte caſus pro maximo haberi nequit; cum enim fractio inverſa $\frac{Q}{P}$ iiſdem caſibus fiat minimum, quibus propoſita $\frac{P}{Q}$ fit maximum, deberet fractio $\frac{Q}{P}$ fieri minimum, ſi Q evaneſcit; hoc autem non ſemper evenit, propterea quod adhuc minores valores, negativos ſcilicet

cet, induere poſſet. Hoc igitur dubio exemto, ſimul regula ante data confirmatur, quod maxima & minima ex aequatione $\frac{dy}{dx} = 0$ elici debeant. Fiet ergo caſu propoſito $\frac{dy}{dx} = \frac{QdP - PdQ}{QQdx}$; ideoque $QdP - PdQ = 0$; huiusque aequationis radices efficient functionem y vel maximum vel minimum. Atque ſi dubium ſit, utrum maximum an minimum locum habeat, confugiendum eſt ad valorem $\frac{ddy}{dx^2}$, qui ſi fuerit affirmativus minimum indicabit, ſin autem ſit negativus maximum. Quod ſi vero & hic valor $\frac{ddy}{dx^2}$ evaneſcat, quod evenit, ſi aequatio $\frac{dy}{dx} = 0$ habeat duas pluresve radices aequales; perpetuo tenendum eſt radices aequales numero pares neque maximum neque minimum producere.

EXEMPLUM I.

Invenire caſus, quibus functio $\frac{x}{1+xx}$ ſit maximum vel minimum.

Primum quidem apparet, hanc functionem in nihilum abire caſibus tribus $x = \infty$, $x = 0$ & $x = -\infty$, unde ad minimum duo recipiet ſive maxima ſive minima. Ad quae invenienda ponatur $y = \frac{x}{1+xx}$; eritque $\frac{dy}{dx} = \frac{1-xx}{(1+xx)^2}$ & $\frac{ddy}{dx^2} = -\frac{6x - 2x^3}{(1+xx)^3}$. Iam ſtatuatur $\frac{dy}{dx} = 0$, erit $1 - xx = 0$, & vel $x = +1$ vel $x = -1$. Priori caſu $x = +1$ fit $\frac{ddy}{dx^2} = -\frac{4}{2^3}$; ideoque y maxim. $= \frac{1}{2}$, poſteriori $x = -1$ fit $\frac{ddy}{dx^2} = +\frac{4}{2^3}$; ideoque y minim. $= -\frac{1}{2}$.

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Haec quoque facilius inveniuntur, ſi fractio propoſita $\frac{x}{1+xx}$ invertatur, ponendo $y = \frac{1+xx}{x} = x + \frac{1}{x}$, dummodo recordemur tum quae maxima inveniuntur, in minima & viciſſim tranſmutari debere. Erit autem $\frac{dy}{dx} = 1 - \frac{1}{xx}$, & $\frac{ddy}{dx^2} = \frac{2}{x^3}$. Statuto ergo $\frac{dy}{dx} = 0$, fit $xx - 1 = 0$, indeque vel $x = +1$ vel $x = -1$ ut ante. Atque caſu $x = +1$ fit $\frac{ddy}{dx^2} = 2$, ideoque y minimum, & formula propoſita $\frac{1}{y}$ maximum. Caſu autem $x = -1$, fit $\frac{ddy}{dx^2} = -2$, unde y maximum & $\frac{1}{y}$ minimum.

EXEMPLUM II.

Invenire caſus quibus formula $\frac{2-3x+xx}{2+3x+xx}$ *fit maximum vel minimum.*

Poſito $y = \frac{xx-3x+2}{xx+3x+2}$, erit $\frac{dy}{dx} = \frac{6x^2-12}{(xx+3x+2)^2}$ & $\frac{ddy}{dx^2} = \frac{-12x^3+72x+72}{(xx+3x+2)^3}$. Statuatur $\frac{dy}{dx} = 0$, fiet vel $x = +\sqrt{2}$ vel $x = -\sqrt{2}$. Priori caſu $x = +\sqrt{2}$, erit $\frac{ddy}{dx^2} = \frac{48\sqrt{2}+72}{(4+3\sqrt{2})^3}$, ideoque affirmativum, ob denominatorem affirmativum; hinc erit y minimum $= \frac{4-3\sqrt{2}}{4+3\sqrt{2}} = 12\sqrt{2} - 17 = -0,02943725$. Poſteriori caſu

$x =$

$x = -\sqrt{2}$ fit $\frac{ddy}{dx^2} = \frac{-48\sqrt{2}+72}{(4-3\sqrt{2})^3} = \frac{24(3-2\sqrt{2})}{(4-3\sqrt{2})^3}$, cuius valor ob numeratorem affirmativum & denominatorem negativum erit negativus, ideoque y fiet maximum $= \frac{4+3\sqrt{2}}{4-3\sqrt{2}} = -12\sqrt{2}-17 = -33,97056274$. Qui valor etſi minor eſt quam prior minimus, tamen ideo eſt maximus, quod maior ſit contiguis proximis, qui oriuntur, ſi loco x vel aliquantillum maiores vel minores valores quam $-\sqrt{2}$ ſubſtituantur. Cum igitur $\sqrt{2}$ inter limites $\frac{4}{3}$ & $\frac{3}{2}$ contineatur, probatio facile inſtituetur hoc modo:

ſi $x = \frac{4}{3}$ fit $y = -\frac{2}{70} = -0,0285$

ſi $x = \sqrt{2}$ fit $y = 12\sqrt{2}-17 = -0,0294$ minimum

ſi $x = \frac{3}{2}$ fit $y = -\frac{1}{35} = -0,0285$

ſi $x = -\frac{4}{3}$ fit $y = -35$

ſi $x = -\sqrt{2}$ fit $y = -33,979$ maximum

ſi $x = -\frac{3}{2}$ fit $y = -35$.

EXEMPLUM III.

Invenire caſus, quibus formula $\frac{xx-x+1}{xx+x-1}$ *fit maximum vel minimum.*

Ponatur $y = \frac{xx-x+1}{xx+x-1}$, eritque $\frac{dy}{dx} = \frac{2xx-4x}{(xx+x-1)^2}$ & $\frac{ddy}{dx^2} = \frac{-4x^3+12xx+4}{(xx+x-1)^3}$. Statuatur $\frac{dy}{dx} = 0$, erit vel

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$x = 0$ vel $x = 2$: priori caſu fit $\frac{ddy}{dx^2} = \frac{4}{-1}$, ideoque erit y maximum $= -1$. Poſteriori caſu $x = 2$ fit $\frac{ddy}{dx^2} = \frac{20}{5^3}$, ideoque y minimum $= \frac{3}{5}$, etiamſi illud maximum minus ſit quam hoc minimum. Probatio patebit ex his poſitionibus:

ſi $x = -\frac{1}{3}$; erit $y = -\frac{15}{11}$

ſi $x = 0$; $y = -1$ maximum

ſi $x = +\frac{1}{3}$; $y = -\frac{7}{5}$

ſi $x = 2 - \frac{1}{3}$; erit $y = \frac{19}{31}$

ſi $x = 2$; $y = \frac{3}{5}$ minimum

ſi $x = 2 + \frac{1}{3}$; $y = \frac{37}{61}$

Quod autem, ſi ponatur $x = 1$, fiat $y = 1$, ideoque > -1, cauſa eſt, quod inter valores ipſius x, 0 & 1 contineatur unus, quo fit $y = \infty$.

EXEMPLUM IV.

Quaerantur caſus, quibus haec fractio $\frac{x^3 + x}{x^4 - xx + 1}$ *fiat maxima vel minima.*

Poſito $y = \frac{x^3 + x}{x^4 - xx + 1}$, erit $\frac{dy}{dx} = \frac{-x^6 - 4x^4 + 4xx + 1}{(x^4 - xx + 1)^2}$ & $\frac{ddy}{dx^2} = \frac{2x^9 + 18x^7 - 30x^5 - 16x^3 + 12x}{(x^4 - xx + 1)^3}$. Habebimus ergo hanc aequationem: $x^6 + 4x^4 - 4xx - 1 = 0$, quae reſolvitur in has duas: $xx - 1 = 0$ & $x^4 + 5x^2 + 1 = 0$, qua-

quarum illius radices ſunt $x = +1$ & $x = -1$; haec vero reſoluta dat $xx = \frac{-5 \pm \sqrt{21}}{2}$, ex qua nulla radix realis emergit. Duarum igitur radicum inventarum prior $x = +1$ facit $\frac{ddy}{dx^2} = -14$, ac propterea y maximum $= 2$; altera radix $x = -1$ facit $\frac{ddy}{dx^2} = +14$ ac propterea y minimum $= -2$.

EXEMPLUM V.

Invenire caſus, quibus haec fractio $\frac{x^3 - x}{x^4 - xx + 1}$ *fit maximum vel minimum.*

Poſito $y = \frac{x^3 - x}{x^4 - xx + 1}$, erit $\frac{dy}{dx} = \frac{-x^6 + 2x^4 + 2x^2 - 1}{(x^4 - x^2 + 1)^2}$ & $\frac{ddy}{dx^2} = \frac{2x^9 - 6x^7 - 18x^5 + 20x^3}{(x^4 - x^2 + 1)^3}$. Facto autem $\frac{dy}{dx} = 0$, erit $x^6 - 2x^4 - 2x^2 + 1 = 0$, quae diviſa per $xx + 1$ dat $x^4 - 3x^2 + 1 = 0$, haecque ulterius reſolvitur in $xx - x - 1 = 0$ & $xx + x - 1 = 0$, unde ſequentes quatuor oriuntur radices reales:

I. $x = \frac{1 + \sqrt{5}}{2}$; II. $x = \frac{1 - \sqrt{5}}{2}$.

III. $x = \frac{-1 + \sqrt{5}}{2}$; IV. $x = \frac{-1 - \sqrt{5}}{2}$.

Quae cum omnes in aequatione $x^4 - 3xx + 1 = 0$ contineantur, poſito $x^4 = 3xx - 1$ fiet pro omnibus

$$\frac{ddy}{dx^2} = \frac{2x(10 - 20xx)}{8x^6} = \frac{5(1 - 2xx)}{2x^5} = \frac{5(1 - 2xx)}{2x(3xx - 1)}$$

&

$$\& \quad y = \frac{x^3 - x}{2xx} = \frac{xx - 1}{2x}.$$

Pro duabus autem prioribus ex aequatione $xx = x + 1$ ortis erit $\frac{ddy}{dx^2} = \frac{-5(2x+1)}{2x(3x+2)} = \frac{-5(2x+1)}{2(5x+3)}$, & $y = \frac{1}{2}$.

Prima igitur radix $x = \frac{1+\sqrt{5}}{2}$ dat $\frac{ddy}{dx^2} = \frac{-5(2+\sqrt{5})}{11+5\sqrt{5}}$, ideoque eſt y maximum. Secunda radix $x = \frac{1-\sqrt{5}}{2}$; dat $\frac{ddy}{dx^2} = \frac{-5(2-\sqrt{5})}{11-5\sqrt{5}} = \frac{-5(\sqrt{5}-2)}{5\sqrt{5}-11}$; ideoque $y = \frac{1}{2}$ erit quoque maximum. Duae reliquae radices dant $y = -\frac{1}{2}$ minimum.

266. In his igitur exemplis exploratio, utrum valor quiſpiam inventus maximum an minimum producat, facilius inſtitui poterit: cum enim ſit $\frac{dy}{dx} = 0$, valor termini $\frac{ddy}{dx^2}$, eius aequationis ratione habita, ſimplicius exprimi poterit. Sit enim propoſita fractio $y = \frac{P}{Q}$; cum ſit $dy = \frac{QdP - PdQ}{QQ}$, & $QdP - PdQ = 0$; erit $ddy = \frac{d(QdP - PdQ)}{Q^2} - \frac{2dQ(QdP - PdQ)}{Q^3}$. At vero ob $QdP - PdQ = 0$, hic poſterior terminus evaneſcit, eritque $ddy = \frac{d.(QdP - PdQ)}{QQ} = \frac{QddP - PddQ}{Q^2}$. Quoniam vero iudicium ex huius termini valore ſive affirmativo ſive negativo petitur, denominator autem Q^2 perpetuo ſit affirma-

affirmativus; ex ſolo numeratore negotium ita confici poterit, ut quoties $Q\,dd\,P - P\,dd\,Q$ ſeu $\frac{d(Q\,dP - P\,dQ)}{dx^2}$ fuerit affirmativum, minimum pronuncietur, ſin ſit negativum maximum. Sive poſtquam inventum fuerit $\frac{dy}{dx}$, cuius forma erit huiuſmodi $\frac{R}{QQ}$, tantum quaeratur $\frac{dR}{dx}$; & quae radix huic expreſſioni valorem affirmativum inducit, ex ea proveniet minimum, & contra maximum.

267. Si denominator fractionis propoſitae fuerit quadratum ſeu altior poteſtas quaecunque, ita ut ſit $y = \frac{P}{Q^n}$, fiet $dy = \frac{Q\,dP - nP\,dQ}{Q^{n+1}}$, & poſito $\frac{Q\,dP - nP\,dQ}{dx} = R$, erit $\frac{dy}{dx} = \frac{R}{Q^{n+1}}$; & maxima minimaque determinabuntur ex radicibus aequationis $R = 0$. Cum deinde ſit $\frac{ddy}{dx^2} = \frac{Q\,dR - (n+1)R\,dQ}{Q^{n+2}}$, ob $R = 0$, fiet $\frac{ddy}{dx} = \frac{dR}{Q^{n+1}}$; cuius valor affirmativus indicabit minimum, negativus autem maximum. Perſpicuum autem eſt, ſi n fuerit numerus impar, ob Q^{n+1} ſemper affirmativum, iudicium ex ſolo $\frac{dR}{dx}$ perfici poſſe; ſin autem n ſit numerus par, adhibeatur formula $\frac{Q\,dR}{dx}$. Ponamus autem porro proponi huiuſmodi fractionem $\frac{P^m}{Q^n} = y$, erit $dy = \frac{(mQ\,dP - nP\,dQ)P^{m-1}}{Q^{n+1}}$; ſi itaque ponatur

$mQ\,dP$

$\frac{mQdP - nPdQ}{dx} = R$, aequationis $R = 0$ radices indicabunt caſus, quibus functio y fit vel maximum vel minimum. Cum igitur ſit $\frac{dy}{dx} = \frac{P^{m-1}R}{Q^{n+1}}$, erit:

$$\frac{ddy}{dx} = \frac{P^{m-2}R[(m-1)QdP - (n+1)PdQ]}{Q^{n+2}} + \frac{P^{m-1}dR}{Q^{n+1}},$$

& ob $R = 0$, fiet $\frac{ddy}{dx^2} = \frac{P^{m-1}dR}{Q^{n+1}dx}$; quae inſuper per quodcunque quadratum $\frac{P^{2\mu}}{Q^{2\nu}}$ dividi poteſt, ad iudicium abſolvendum. Praeterea vero quoque aequatio $P = 0$ dabit maximum vel minimum, ſi m fuerit numerus par; atque ſimili modo formulam inverſam $\frac{Q^n}{P^m}$ ſpectando, prodibit maximum vel minimum ponendo $Q = 0$, ſi n fuerit numerus par, uti ſupra §. 257. oſtendimus: hic autem ad maxima vel minima hinc oriunda non reſpicimus, ſed tantum ad uſum methodi explicandum ea indagamus, quae oriuntur ex aequatione $R = 0$.

EXEMPLUM I.

Proponatur fractio $\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$, quae, quo caſu fiat maximum vel minimum, quaeritur.

Poſito $y = \frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$, primo quidem patet fore $y = 0$ ſi $x = -\frac{\alpha}{\beta}$, & $y = \infty$ ſi $x = -\frac{\gamma}{\delta}$: quorum caſuum ille dabit minimum, hic vero maximum, ſi m & n fuerint numeri pares. Praeterea vero erit:

$$\frac{dy}{dx} = \frac{(\alpha+\beta x)^{m-1}}{(\gamma+\delta x)^{n+1}}[(m-n)\beta\delta x + m\beta\gamma - n\alpha\delta],$$ ideoque

que $R = (m - n)\,\beta\delta x + m\beta\gamma - n\alpha\delta$. Quare poſito $R = 0$, erit $x = \frac{n\alpha\delta - m\beta\gamma}{(m-n)\beta\delta}$. Deinde ob $\frac{dR}{dx} = (m - n)\,\beta\,\delta$, diſpiciendum eſt, utrum

$$\frac{P^{m-1}\,dR}{Q^{n+1}\,dx} = \frac{m^{m-1}\beta^{n+1}}{n^{n+1}\delta^{m-1}}\left(\frac{\alpha\delta-\beta\gamma}{m-n}\right)^{m-n-2}\frac{dR}{dx}$$

ſit quantitas affirmativa an negativa; priori caſu formula propoſita erit minimum., poſteriori maximum. Sic ſi fuerit $y = \frac{(x+3)^3}{(x+2)^2}$, fiet $\frac{P^{m-1}\,dR}{Q^{n+1}\,dx} = \frac{9}{8}$, ideoque formula $\frac{(x+3)^3}{(x+2)^2}$ fiet minimum, ſi ponatur $x = 0$. Sin autem ſit $y = \frac{(x-1)^m}{(x+1)^n}$, erit $\frac{P^{m-1}\,dR}{Q^{n+1}\,dx} = \frac{m^{m-1}}{n^{n+1}}\left(\frac{n-m}{2}\right)^{n-m+2}(m-n)$, & $x = \frac{n+m}{n-m}$. Cum autem m & n ponantur numeri affirmativi, iudicium petendum erit ex formula $(n-m)^{n-m+2}(m-n)$, ſeu $(n-m)^{n-m}(m-n)$. Si igitur fuerit $n > m$, valor erutus $x = \frac{n+m}{n-m}$ ſemper dabit maximum; ſin autem ſit $n < m$, numerus $m - n$ par dabit minimum, at impar, maximum: ſic $\frac{(x-1)^3}{(x+1)^2}$ fiet maximum poſito $x = -5$; fit enim $y = -\frac{6^3}{4^2} = -\frac{27}{4}$.

EXEMPLUM II.

Proponatur formula $y = \frac{(1+x)^3}{(1+xx)^2}$.

Erit $\frac{dy}{dx} = \frac{(1+x)^2}{(1+xx)^3}\,(3 - 4x - xx)$,

$$\& \quad \frac{P^{m-1}}{Q^{n+1}}.\frac{dR}{dx} = -\frac{(1+x)^2}{(1+xx)^3}(2x+4);$$

ubi cum $(1+x)^2$ & $(1+xx)^3$ ſemper habeant valorem affirmativum, iudicium relinquetur formulae $-x-2$, quae ſi fuerit affirmativa minimum, ſin negativa maximum indicat. At vero ex aequatione $3-4x-xx=0$ ſequitur vel $x=-2+\surd 7$ vel $x=-2-\surd 7$. Priori caſu fit $-x-2=-\surd 7$, ideoque fractio propoſita erit maximum, poſteriori vero caſu minimum ob $-x-2=+\surd 7$. Poſito autem $x=-2+\surd 7$, erit $1+x=-1+\surd 7$, & $1+xx=12-4\surd 7$, unde

$$y=\left(\frac{-1+\surd 7}{12-4\surd 7}\right)^2(\surd 7-1)=\frac{(2+\surd 7)^2(\surd 7-1)}{16}=\frac{17+7\surd 7}{16}$$

$$=2,220.$$

Poſito autem $x=-2-\surd 7$ fiet $y=\frac{17-7\surd 7}{16}=-0,0950$

268. Dantur etiam functiones irrationales & tranſcendentes, quae proprietatem functionum uniformium habent, & hancobrem maxima & minima eodem modo inveniri poſſunt. Radices enim cubicae & omnium imparium poteſtatum revera ſunt uniformes, cum nonniſi unicum valorem realem exhibeant: radices autem quadratae atque omnium poteſtatum parium, etſi revera, quoties ſunt reales, geminum valorem indicant, alterum affirmativum alterum negativum, tamen unuſquiſque ſeorſim ſpectari poteſt, hocque ſenſu etiam maxima & minima inveſtigari poſſunt. Sic ſi y fuerit functio quaecunque ipſius x, etſi $\surd y$ geminum habet valorem tamen uterque ſeorſim tractari poterit. Scilicet $+\surd y$ maximum vel minimum habebit valorem, ſi y talem habuerit, dummodo fuerit affirmativus; quia alioquin $\surd y$ evaderet imaginarium. Vice verſa autem $-\surd y$ fiet minimum vel maximum, iiſdem caſibus, quibus $+\surd y$ fit maximum vel minimum. Poteſtas autem quaecunque $y^{\frac{m}{n}}$, iiſdem caſibus fiet maximum vel minimum, ſiquidem n fuerit numerus im-

par;

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par; at ſi n fuerit numerus par, ii tantum caſus valent, quibus y induit valorem affirmativum: hiſque caſibus ob ancipitem valorem gemina prodibunt maxima vel minima.

269. Quoniam aequatio differentialis, quae ex poteſtate functionis y^m naſcitur, eſt $\frac{y^{m-1}\,dy}{dx} = 0$, cuius radices ſimul caſus, quibus poteſtas ſurda $y^{\frac{m}{n}}$ fit maximum vel minimum, indicant, at hoc indagandum duplex habetur aequatio, altera $y^{m-1} = 0$, altera $\frac{dy}{dx} = 0$, quarum illa abit in $y = 0$, atque tum ſolum maxima & minima exhibet, ſi $m - 1$ fuerit numerus impar, ſeu ſi m fuerit numerus par, ob rationes §. 257. allegatas. Quare cum n ſit numerus impar, ſi m fuerit numerus par, ſi numeros pares per 2μ & impares per $2\nu - 1$ indicemus, functio $y^{2\mu:(2\nu-1)}$ evadet maxima vel minima tribuendis ipſi x valoribus, quos tam ex hac aequatione $y = 0$ quam ex hac $\frac{dy}{dx} = 0$ adipiſcitur. Sin autem m ſit numerus impar, functio $y^{(2\mu-1):(2\nu-1)}$, vel $y^{(2\mu-1):2\nu}$ tum ſolum fit maxima vel minima, cum loco x ſubſtituitur valor ex hac aequatione $\frac{dy}{dx} = 0$. Ac poſteriori quidem caſu $y^{(2\mu-1):2\nu}$ maxima & minima tantum proveniunt, ſi y ab inventis ex aequatione $\frac{dy}{dx} = 0$ valoribus, affirmativos recipiat valores.

270. Sic iſta formula $x^{\frac{2}{3}}$ fit minimum, ponendo $x = 0$, propterea quod hoc caſu x^2 fit minimum. Niſi autem

tem formulam $x^{\frac{2}{3}}$ ad formam x^2 reducamus, methodus ante tradita hoc minimum indicaret; propterea quod caſu $x = 0$, termini ſeriei $y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} +$ &c. unde iudicium peti debet, praeter primum omnes fiunt infiniti. Facto enim $y = x^{\frac{2}{3}}$ erit:

$$\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}}; \quad \frac{ddy}{dx^2} = \frac{-2}{9x^{\frac{4}{3}}}; \quad \frac{d^3y}{dx^3} = \frac{2.4}{27x^{\frac{7}{3}}}; \quad \&c.$$

Hinc neque aequatio $\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}} = 0$ oſtendit valorem $x = 0$, neque termini ſequentes rationem maximi minimive indicant. Cum igitur aſſumſimus ſeriem $y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} +$ &c. fieri convergentem, ſi ω ſtatuatur quantitas valde parva; ii caſus utique methodum generalem effugiunt, quibus haec ſeries fit divergens, quod evenit exemplo hic allato $y = x^{\frac{2}{3}}$, ſi ponatur $x = 0$. Quamobrem his caſibus ea reductione, qua ante uſi ſumus, erit opus, quo expreſſio propoſita ad aliam formam revocetur, quae huic incommodo non ſit ſubiecta. Hoc autem tantum pauciſſimis caſibus uſu venit, qui in formula $y^{\frac{2\mu}{2\nu - 1}}$ continentur, vel ad eam facile reducuntur. Sic ſi requirantur maxima minimave formulae $y^{\frac{2\mu}{2\nu - 1}} z$, exiſtente z functione quacunque ipſius x, inveſtigetur forma haec $y^{2\mu} z^{2\nu - 1}$, quippe quae iiſdem caſibus fit maxima vel minima, quibus ipſa propoſita.

271. Hoc caſu excepto, qui iam facile expeditur, functio-

ctiones, quae continent quantitates irrationales, eodem modo quo rationales tractari, earumque maxima & minima determinari poſſunt, id quod ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Propoſita ſit formula $\sqrt{(aa+xx)}-x$, *quae quibus caſibus fiat maxima vel minima, quaeritur.*

Poſito $y=\sqrt{(aa+xx)}-x$, erit $\frac{dy}{dx}=\frac{x}{\sqrt{(aa+xx)}}-1$, & $\frac{ddy}{dx^2}=\frac{aa}{(aa+xx)^{3:2}}$. Facto ergo $\frac{dy}{dx}=0$, erit $x=\sqrt{(aa+xx)}$, ideoque $x=\infty$, ac fit $\frac{ddy}{dx^2}=0$. Simili vero modo fiunt ſequentes termini $\frac{d^3y}{dx^3}$; $\frac{d^4y}{dx^4}$; &c. omnes $=0$; ex quo iudicium incertum relinquitur, utrum ſit maximum an minimum, tam ſi fiat $x=-\infty$, quam $x=+\infty$: Interim ponendo $x=+\infty$, ob $\sqrt{(aa+xx)}=x+\frac{aa}{2x}$, fit $y=0$, qui valor omnium eſt minimus.

EXEMPLUM II.

Quaerantur caſus, quibus haec forma $\sqrt{(aa+2bx+mxx)}-nx$ *ſit maximum vel minimum.*

Poſito $y=\sqrt{(aa+2bx+mxx)}-nx$, erit: $\frac{dy}{dx}=\frac{b+mx}{\sqrt{(aa+2bx+mxx)}}-n$, quo facto $=0$, erit: $bb+2mbx+mmxx=nnaa+2nnbx+mnnxx$, ſeu $xx=\frac{2bx(nn-m)+nnaa-bb}{mm-mnn}$, ideoque

$$x=\frac{(nn-m)b\pm\sqrt{[mnn(m-nn)aa-nn(m-nn)bb]}}{m(m-nn)},$$

ſive $x=-\frac{b}{m}\pm\frac{n}{m}\sqrt{\frac{maa-bb}{m-nn}}$: unde fit

$\sqrt{(a}$

$$\sqrt{(aa+2bx+mxx)} = \frac{b+mx}{n} = \pm\sqrt{\frac{maa-bb}{m-nn}}.$$

Cum igitur ſit $\frac{ddy}{dx^2} = \frac{maa-bb}{(aa+2bx+mxx)^{\frac{3}{2}}}$, erit:

$$\frac{ddy}{dx^2} = \frac{maa-bb}{\pm\left(\frac{maa-bb}{m-nn}\right)^{\frac{3}{2}}} = \frac{\pm(m-nn)\sqrt{(m-nn)}}{\sqrt{(maa-bb)}}.$$

Niſi ergo fuerit $\frac{m-nn}{maa-bb}$ quantitas affirmativa, maximum minimumve plane non datur. Sin autem ſit quantitas affirmativa, ſignum ſuperius dabit minimum ſi $m > nn$, maximum vero ſi $m < nn$: contrarium evenit, ſi ſignum inferius valeat. Si ergo ſit $m=2$, $n=1$, & $b=0$, formula $\sqrt{(aa+2xx)}-x$ fit minimum ponendo $x = +\frac{1}{2}\sqrt{2aa} = \frac{a}{\sqrt{2}}$, at maximum ponendo $x = -\frac{a}{\sqrt{2}}$. Erit ergo minimum $= a\sqrt{2} - \frac{a}{\sqrt{2}} = \frac{a}{\sqrt{2}}$, & maximum $= a\sqrt{2} + \frac{a}{\sqrt{2}} = \frac{3a}{\sqrt{2}}$.

EXEMPLUM III.

Quaerantur caſus, quibus haec expreſſio

$$\sqrt[4]{(1+mx^4)} + \sqrt[4]{(1-nx^4)}$$

fiat maximum vel minimum.

Cum ſit $\frac{dy}{dx} = \frac{mx^3}{(1+mx^4)^{\frac{3}{4}}} - \frac{nx^3}{(1-nx^4)^{\frac{3}{4}}}$, fiet $mx^3(1-nx^4)^{\frac{3}{4}} = nx^3(1+mx^4)^{\frac{3}{4}}$, ideoque $m^4(1-nx^4)^3 = n^4(1+mx^4)^3$, ſeu $n^4 - m^4 + 3mn(n^3+m^3)x^4 + 3m^2n^2(n^2-m^2)x^8 + m^3n^3(n+m)x^{12} = 0$. Niſi ergo haec aequatio radicem poſitivam habeat pro x^4, maxi-

maximum minimumve prorſus non datur. Quia haec aequatio generaliter commode reſolvi nequit, fiet enim

$$x^4 = \frac{m^{\frac{4}{3}} - n^{\frac{4}{3}}}{mn(\sqrt[3]{m} + \sqrt[3]{n})} \text{ ſeu } x^4 = \frac{m - \sqrt[3]{m^2 n} + \sqrt[3]{mn^2} - n}{mn}.$$

ponamus pro caſu ſpeciali $m = 8n$, eritque

$$-4095 + 24.513nx^4 - 3.63.64n^2x^8 + 9.512n^3x^{12} = 0$$

ſeu $512\,n^3x^{12} - 1344\,n^2x^8 + 1368\,nx^4 - 455 = 0$

ponatur $8nx^4 = z$, erit $z^3 - 21z^2 + 171z - 455 = 0$ quae diviſorem habet $z - 5$, alterque factor erit: $zz - 16z + 91 = 0$ radices continens imaginarias. Erit ergo tantum $z = 8nx^4 = 5$, ideoque $x = \sqrt[4]{\frac{5}{8n}}$ qui valor reddet expreſſionem $\sqrt[4]{(1 + 8nx^4)} + \sqrt[4]{(1 - nx^4)}$ maximum vel minimum. Quorum utrum eveniat, quaeratur $\frac{ddy}{dx^2} = \frac{3mxx}{(1 + nx^4)^{\frac{7}{4}}} - \frac{3nxx}{(1 - nx^4)^{\frac{7}{4}}}$. At ob $m = 8n$, poſito $x^4 = \frac{5}{8n}$, erit $\frac{ddy}{dx^2} = \left(\frac{24n}{6^{\frac{7}{4}}} - \frac{3n}{(3:8)^{\frac{7}{4}}}\right)xx = -\frac{360nxx}{6^{\frac{7}{4}}}$ ideoque negativum, ergo fiet $\sqrt[4]{(1 + 8nx^4)} + \sqrt[4]{(1 - nx^4)}$ maximum poſito $x = \sqrt[4]{\frac{5}{8n}}$. Erit vero hoc maximum $= \sqrt[4]{6} + \sqrt[4]{\frac{3}{8}} = \frac{3\sqrt[4]{6}}{2}$. Si loco nx^4 ponamus u, patet hanc expreſſionem $\sqrt[4]{(1 + 8u)} + \sqrt[4]{(1 - u)}$ fieri maximam, poſito $u = \frac{5}{8}$, huncque valorem maximum

mum fore $= \frac{3\sqrt[4]{6}}{2} = 2,347627$. Quicunque ergo valor praeter $\frac{5}{8}$ pro u ſcribatur, expreſſio minorem accipiet valorem.

272. Simili modo maxima ac minima determinabuntur, ſi quantitates quoque tranſcendentes in expreſſione propoſita inſint. Niſi enim functio propoſita fuerit multiformis, atque aliquot eius ſignificatus ſimul conſiderari debeant, radices aequationis differentialis oſtendent maxima vel minima, niſi affuerint radices aequales, quarum numerus ſit par. Hanc ergo inveſtigationem in aliquot exemplis declarabimus.

EXEMPLUM I.

Invenire numerum, qui ad ſuum logarithmum minimam teneat rationem.

Dari huiuſmodi rationem minimam $\frac{x}{lx}$ inde patet, quod haec ratio tam poſito $x=1$, quam $x=\infty$ fiat infinita. Viciſſim ergo habebit fractio $\frac{lx}{x}$ alicubi maximum valorem, eodem ſcilicet caſu, quo $\frac{x}{lx}$ fit minimum. Ad hunc caſum indagandum ponatur $y=\frac{lx}{x}$, fietque $\frac{dy}{dx}=\frac{1}{xx}-\frac{lx}{xx}$. Quo nihilo aequali poſito erit $lx=1$, & quia hic logarithmum hyperbolicum aſſumſimus, ſi e ponatur numerus cuius logarithmus hyperbolicus ſit $=1$, erit $x=e$. Cum igitur omnes logarithmi ad hyperbolicos in data ſint ratione, erit in quocunque logarithmorum canone $\frac{e}{le}$ minimum, ſeu $\frac{le}{e}$ maximum. Quoniam in logarithmis tabularibus eſt

$le = 0,4342944819$, fractio $\frac{lx}{x}$ perpetuo erit minor quam $\frac{4342944819}{27182818284}$, ſeu proxime quam $\frac{47}{305}$ neque ullus datur numerus, qui ad ſuum logarithmum minorem teneat rationem quam 305 ad 47. Eſſe autem hoc caſu $\frac{lx}{x}$ maximum inde patet, quod ob $\frac{dy}{dx} = \frac{1 - lx}{xx}$ fiat $\frac{ddy}{dx^2} = -\frac{1}{x^3} - \frac{2(1-lx)}{x^3} = -\frac{1}{x^3}$, propter $1 - lx = 0$, ideoque negativum.

EXEMPLUM II.

Invenire numerum x, *ut haec poteſtas* $x^{1:x}$ *fiat maximum.*

Dari huius formulae valorem maximum inde patet, quod numeris loco x ſubſtituendis ſit

$$1^{1:1} = 1,000000$$
$$2^{1:2} = 1,414213$$
$$3^{1:3} = 1,442250$$
$$4^{1:4} = 1,414213$$

Ponatur ergo $x^{1:x} = y$, eritque $\frac{dy}{dx} = x^{1:x}\left(\frac{1}{xx} - \frac{lx}{xx}\right)$. Quo valore nihilo aequali poſito, erit $lx = 1$ & $x = e$, exiſtente $e = 2,718281828$. Et cum ſit $\frac{dy}{dx} = (1 - lx)\frac{x^{1:x}}{xx}$, erit $\frac{ddy}{dx^2} = \frac{-x^{1:x}}{x^3} + (1 - lx)d.\frac{x^{1:x}}{xx} = -\frac{x^{1:x}}{x^3}$ ob $1 - lx = 0$. Quare cum ſit $\frac{ddy}{dx^2}$ quantitas negativa, fiet $x^{1:x}$ maximum caſu $x = e$. Cum autem ſit $e = 2,718281828$, reperitur fore $e^{\frac{1}{e}} = 1,444667861009764$, qui valor obtinetur facile ex ſerie $e^{\frac{1}{e}} = 1 + \frac{1}{e} + \frac{1}{2e^2} + \frac{1}{6e^3} + \frac{1}{24e^4} +$ &c.

Hoc exemplum quoque ex praecedenti reſolvitur: ſi enim ſit $x^{1:x}$ maximum, quoque eius logarithmus, qui eſt $\frac{lx}{x}$, debebit eſſe maximum; quod quo fiat, debet eſſe $x = e$, uti invenimus.

EXEMPLUM III.

Invenire arcum x, *ut ſit eius ſinus maximus vel minimus.*

Poſito $\sin x = y$ erit $\frac{dy}{dx} = \cos x$, ideoque $\cos x = 0$, unde prodeunt ſequentes valores pro $x: \pm\frac{\pi}{2}; \pm\frac{3\pi}{2}; \pm\frac{5\pi}{2}$ &c. Fit autem $\frac{ddy}{dx^2} = -\sin x$. Cum igitur hi valores pro x ſubſtituti dent pro $\sin x$ vel $+1$ vel -1, illi erunt maximi, hi vero minimi, uti conſtat.

EXEMPLUM IV.

Invenire arcum x, *ut rectangulum* x. *ſin* x *fiat maximum.*

Dari maximum inde patet, quod poſito vel $x = 0$ vel $x = 180^\circ$ utroque caſu rectangulum propoſitum evaneſcat. Sit igitur $y = x \sin x$; erit $\frac{dy}{dx} = \sin x + x\cos x$, ideoque $\mathrm{tang}\, x = -x$. Sit $x = 90^\circ + u$, erit $\mathrm{tang}\, x = -\cot u$, ergo $\cot u = 90 + u$. Ad quam aequationem modo ſupra tradito reſolvendam ponatur $z = 90 + u - \cot u$, ſitque f valor arcus u quaeſitus. Cum ſit $dz = du + \frac{du}{\sin u^2}$, erit

$$p = \frac{du}{dz} = \frac{\sin u^2}{1+\sin u^2}; \quad dp = \frac{2du \sin u \cos u}{(1+\sin u)^2}; \quad \text{ideoque}$$

$$\frac{dp}{dz} = q = \frac{2\sin u^3 \cos u}{(1+\sin u^2)^3}; \quad dq = \frac{6du \sin u^2 \cos u^2 - 2du \sin u^4}{(1+\sin u^2)^3} - \frac{12du \sin u^4 \cos u^2}{(1+\sin u^2)^4}.$$

Ergo

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Ergo $\frac{dq}{dz} = r = \frac{6\,\text{ſin}\,u^4\,\text{coſ}\,u^2 - 2\,\text{ſin}\,u^6}{(1+\text{ſin}\,u^2)^4} - \frac{12\,\text{ſin}\,u^6\,\text{coſ}\,u^2}{(1+\text{ſin}\,u^2)^5}$

$$= \frac{6\,\text{ſin}\,u^4 - 14\,\text{ſin}\,u^6 + 4\,\text{ſin}\,u^8}{(1+\text{ſin}\,u^2)^5}.$$

Ex quibus erit $f = u - pz + \frac{1}{2}qzz - \frac{1}{6}rz^3 + \&c.$ Ponatur, poſtquam aliquot tentaminibus proximus ipſius f valor eſt detectus, $u = 26°,\ 15'$, erit $90 + u = 116°,\ 15'$, & arcus cotangenti u aequalis ita definiatur:

$$A\ l\,\text{cot}\,u = 10,3070250$$
$$\text{ſubtrahatur} \quad 4,6855749$$
$$5,6214501$$
$$\text{Ergo} \quad \text{cot}\,u = 418263,\ 7''$$
$$\text{ſeu} \quad \text{cot}\,u = 116°,\ 11',\ 3\tfrac{7}{10}''$$
$$\text{unde} \quad z = 3',\ 56\tfrac{3}{10}'' = 236,\ 3''.$$

Iam ad valorem termini pz inveniendum, iſte inſtituatur calculus:

$$l\,\text{ſin}\,u = 9,6457058$$
$$l\,\text{ſin}\,u^2 = 9,2914116$$
$$1 + \text{ſin}\,u^2 = 1,19561$$
$$l(1 + \text{ſin}\,u^2) = 0,0775895$$
$$lp = 9,2138221$$
$$lz = 2,3734637$$
$$lpz = 1,5872858$$
$$\text{Ergo} \quad pz = 38,6621 \ \text{ſecundis}$$
$$\text{ſeu} \quad pz = 38'',\ 39''',\ 43''''.$$
$$\text{ab} \quad u = 26°,\ 15'$$
$$\text{fiet} \quad f = 26°,\ 14',\ 21'',\ 20''',\ 17''''.$$
$$\&\ \text{arcus quaeſitus} \quad x = 116,\ 14,\ 21,\ 20,\ 17$$

 Ter-

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Tertius vero terminus $\frac{1}{2}qzz = \frac{\text{ſin}\,u^3\ \text{coſ}\,u}{(1+\text{ſin}\,u^2)^3}zz$ inſuper addi debet.

Cuius valor ut inveniatur, unum z in partibus radii exprimi debet hoc modo:

$$l\,z'' = 2,3734637$$

add. $4,6855749$

$$7,0590386$$

add. $l\,\frac{\text{ſin}\,u^2}{1+\text{ſin}\,u^2}z = 1,5872858$

$$8,6463244$$

add. $l\,\text{ſin}\,u = 9,6457058$

$l\,\text{coſ}\,u = 9,9527308$

$$8,2447600$$

ſubtr. $l\,(1+\text{ſin}\,u^2)^3 = 0,1551790$

$$l\,\tfrac{1}{2}qzz = 8,0895810$$

Ergo $\frac{1}{2}qzz = 0,012291$

ſeu $\frac{1}{2}qzz = 44'''',\ 15'''''$.

Unde & hoc termino adhibito fiet arcus quaeſitus

$$x = 116^\circ,\ 14',\ 21'',\ 21''',\ 0''''$$

maioribus autem logarithmis adhibitis reperitur

$$x = 116^\circ,\ 14',\ 21'',\ 20''',\ 35'''',\ 47'''''.$$

CAPUT

CAPUT XI.

DE MAXIMIS ET MINIMIS FUNCTIONUM MULTIFORMIUM PLURESQUE VARIABILES COMPLECTENTIUM.

273.

Si y fuerit functio multiformis ipſius x, ita ut pro unoquoque valore ipſius x ea plures obtineat valores reales; tum variato x plures illi ipſius y valores ita inter ſe connectentur, ut plures ſeries valorum ſucceſſivorum repraeſentent. Si enim y tanquam applicatam lineae curvae conſideremus, x exiſtente abſciſſa, quot y habuerit valores reales diverſos, totidem diverſi eiuſdem curvae rami eidem abſciſſae x reſpondebunt, atque hinc illi ipſius y valores ſucceſſivi, qui eundem ramum conſtituunt, cohaerere cenſendi ſunt; valores autem ad diverſos ramos relati erunt inter ſe disiuncti. Tot igitur ſeries valorum cohaerentium ipſius y habebimus, quot diverſos valores reales pro quovis ipſius x valore receperit; atque in qualibet ſerie valores ipſius y, dum x creſcens aſſumitur, vel creſcent vel decreſcent, vel poſtquam creverint iterum decreſcent, vel vice verſa. Ex quo perſpicuum eſt, in unaquaque valorum cohaerentium ſerie aeque dari maxima minimave, atque in functionibus uniformibus.

274. Ad haec maxima minimave determinanda eadem quoque methodus valebit, quam capite praecedente pro functionibus uniformibus tradidimus. Cum enim, ſi variabilis x incremento ω augeatur, functio y perpetuo recipiat hanc formam $y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} +$ &c. neceſſe eſt ut caſu maximi minimive terminus $\frac{\omega dy}{dx}$ evaneſcat, fiatque $\frac{dy}{dx} = 0$.

Radi-

Radices ergo huius aequationis $\frac{dy}{dx} = 0$ eos ipſius x valores indicabunt, quibus in ſingulis valorum ipſius y cohaerentium ſeriebus, maxima minimave reſpondeant. Neque vero ambiguum erit, in quanam valorum cohaerentium ſerie detur maximum minimumve. Cum enim in aequatione $\frac{dy}{dx} = 0$ ambae inſint variabiles x & y, valores ipſius x definiri nequeunt, niſi ope aequationis, qua relatio functionis y ab x continetur, variabilis y eliminetur; antequam autem hoc fit, pervenitur ad aequationem, qua valor ipſius y per functionem rationalem ſeu uniformem ipſius x exprimitur. Hinc inventis valoribus ipſius x, cuique reſpondens valor ipſius y reperietur, qui erit maximus vel minimus in ſerie valorum ſucceſſivorum cohaerentium, ad quam pertinet.

275. Iudicium autem, utrum iſti valores ipſius y ſint maximi an minimi inſtituetur eodem modo, quem ante indicavimus. Scilicet quaeratur valor ipſius $\frac{ddy}{dx^2}$ finitis terminis expreſſus, in eoque loco x ſubſtituatur unuſquiſque ipſius x valor inventus ſucceſſive, ſimul autem pro y ponatur valor, qui ipſi pro quolibet ipſius x valore convenit; quo facto diſpiciatur, utrum expreſſio $\frac{ddy}{dx^2}$ adeptura ſit valorem affirmativum an negativum, priorique caſu minimum, poſteriori vero maximum indicabitur. Quodſi vero & $\frac{ddy}{dx^2}$ evaneſcat, tum procedendum erit ad formulam $\frac{d^3y}{dx^3}$, quae ſi eodem caſu non evaneſcat, neque maximum habebitur neque minimum: ſin autem quoque $\frac{d^3y}{dx^3}$ evaneſcat, iudicium formari

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oportebit ex formula $\frac{d^4y}{dx^4}$ eodem modo, quo ratione formulae $\frac{ddy}{dx^2}$ praecipimus. Atque ſi quoque $\frac{d^4y}{dx^4}$ quopiam caſu evaneſcat, ad differentiale quintum ipſius y erit progrediendum: perpetuo autem quouſque progredi neceſſe fuerit, iudicia ex differentialibus ordinum imparium ſimilia ſunt illi, quod de formula $\frac{d^3y}{dx^3}$ dedimus. His ſcilicet caſibus in formulis $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ &c. eouſque erit pergendum, quoad perveniatur ad talem, quae propoſito caſu non evaneſcat; quae ſi fuerit differentialis ordinis imparis, neque maximum neque minimum indicabitur, ſin autem fuerit ordinis paris, eius valor affirmativus minimum, negativus vero maximum innuet.

276. Ponamus functionem y determinari ex x per aequationem quamcunque: quae aequatio ſi differentietur, induet huiuſmodi formam $Pdx + Qdy = 0$. Facto ergo $\frac{dy}{dx} = 0$, erit $\frac{P}{Q} = 0$, ideoque vel $P = 0$ vel $Q = \infty$. Poſterior quidem aequatio, ſi relatio inter x & y exprimatur per aequationem rationalem integram, locum habere nequit; quia vel x vel y vel utramque fieri oporteret infinitam. Quare iudicium relinquetur aequationi $P = 0$, cuius radices, ſeu valores ipſius x, quos adipiſcitur, poſtquam ope aequationis propoſitae variabilis y penitus fuerit eliminata, indicabunt caſus, quibus valores ipſius y fiunt maximi vel minimi. Ad iudicium vero, utrum prodeat maximum an minimum abſolvendum, examinetur formula $\frac{ddy}{dx^2}$. Aequatio vero differentialis $Pdx + Qdy = 0$ denuo differentiata, ſi ponamus

dP

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$dP = Rdx + Sdy$ & $dQ = Tdx + Vdy$, dabit (poſito dx conſtante):

$$R\,dx^2 + S\,dx\,dy + T\,dx\,dy + V\,dy^2 + Q\,ddy = 0.$$

Cum autem iam ſit $\frac{dv}{dx} = 0$, aequatione per dx^2 diviſa fiet $R + \frac{Q\,ddy}{dx^2} = 0$, ideoque $\frac{ddv}{dx^2} = -\frac{R}{Q}$. Hinc in aequatione differentiali $P\,dx + Q\,dy = 0$, differentietur tantum quantitas P, ponendo y conſtans, prodibitque $R\,dx$, tum indagetur, valor fractionis $\frac{R}{Q}$, qui ſi fuerit affirmativus, maximum, ſin negativus minimum indicabit.

277. Sit y functio biformis ipſius x, quae determinetur per hanc aequationem $yy + py + q = 0$, denotantibus p & q functiones quaſcunque ipſius x uniformes. Erit ergo differentiando $2y\,dy + p\,dy + y\,dp + dq = 0$, ideoque $P\,dx = y\,dp + dq$. Poſito igitur $P = 0$ erit $y\,dp + dq = 0$ prodibitque $y = -\frac{dq}{dp}$, ſicque y per functionem ipſius x uniformem exprimitur, ita ut, quicunque valor pro x fuerit inventus, ex eo & y valorem determinatum unicum acquirat. Eliminatio vero nunc ipſius y erit facilis; nam ſi in aequatione propoſita $yy + py + q = 0$ loco y valor $-\frac{dq}{dp}$ ſubſtituatur, habebitur $dq^2 - p\,dp\,dq + q\,dp^2 = 0$, quae aequatio diviſa per dx^2 & reſoluta praebebit valores ipſius x omnes, quibus maxima vel minima reſpondent: quod clarius fiet ſequentibus exemplis.

EXEMPLUM I.

Propoſita aequatione $yy + mxy + aa + bx + nxx = 0$ *definire maxima vel minima functionis* y.

Differentiata aequatione habebimus:

$2ydy$

$$2ydy + mxdy + mydx + bdx + 2nxdx = 0$$

unde fit $P = my + b + 2nx$, & $Q = 2y + mx$. Poſito ergo $P = 0$ fiet $y = \frac{-b - 2nx}{m}$; qui valor in ipſa aequatione ſubſtitutus dat:

$$\begin{matrix} \frac{4nn}{mm}xx + \frac{4nb}{mm}x + \frac{bb}{mm} \\ -2nxx - bx + aa \\ + nxx + bx \end{matrix} = 0$$

ſeu $xx = \frac{4nbx + bb + mmaa}{mmn - 4nn}$; unde fit

$$x = \frac{2nb \pm \sqrt{[mmnbb + mmn(mm-4n)aa]}}{mmn - 4nn}$$

ſeu $x = \frac{2nb \pm m\sqrt{[nbb + n(mm-4n)aa]}}{n(mm-4n)}$

& $y = \frac{-mb \mp 2\sqrt{[nbb + n(mm-4n)aa]}}{mm - 4n}$.

Tum poſito ſolo x variabili fit $dP = 2ndx$, ideoque $R = 2n$. At eſt $Q = 2y + mx = \pm\frac{\sqrt{[nbb + n(mm-4n)aa]}}{n}$, unde $\frac{R}{Q} = \frac{\pm 2nn}{\sqrt{[nbb + n(mm-4n)aa]}}$, cuius numerator $2nn$ cum ſit perpetuo affirmativus, ſi ſignum ſuperius valeat, prodibit pro y valor maximus, ſin inferius prodibit minimus. Ubi ſequentia annotari debent.

I. Si fuerit $m = 0$, ex aequatione $P = 0$ ſtatim ſequitur $x = -\frac{b}{2n}$, ut nulla eliminatione opus ſit. Huicque valori geminus ipſius y reſpondet ob $y = \pm\frac{1}{2n}\sqrt{(nbb - 4nnaa)}$ quorum alter affirmativus eſt maximus, alter negativus minimus.

II. Si ſit $n = 0$, ſit $y = -\frac{b}{m}$ & x in infinitum excreſcit, atque y per ſpatium infinitum eundem valorem retinet, ita ut neque maximus ſit neque minimus.

III. Si ſit $mm = 4n$, erit $4nbx + bb + mmaa = 0$ ſeu $x = \frac{bb + mmaa}{-mmb}$; ſietque $y = \frac{-b - 2nx}{m} =$ $\frac{-2b - mmx}{m} = -\frac{2b}{m} + \frac{bb + mmaa}{mb} = \frac{mmaa - bb}{mb}$. Huic ergo valori ipſius $x = \frac{-mmaa - bb}{mmb}$ alter ipſius y valor, qui reſpondet $\frac{mmaa - bb}{mb}$, erit maximus vel minimus. Quia autem, ut iſte ipſius y valor prodeat, in expreſſione $y = \frac{-mb \mp 2\sqrt{[nbb + n(mm - 4n)aa]}}{mm - 4n}$ ſignum inferius valere debet, erit valor ipſius y minimus.

EXEMPLUM II.

Propoſita aequatione $yy - xxy + x - x^3 = 0$ *definire valores ipſius* y *maximos vel minimos.*

Differentiata aequatione prodit:

$2ydy - xxdy - 2xydx + dx - 3xxdx = 0$. Fitque $P = 1 - 3xx - 2xy$ & $Q = 2y - xx$. Quare poſito $P = 0$, erit $y = \frac{1 - 3xx}{2x}$; ideoque hoc valore ſubſtituto:

$$\frac{1}{4xx} - \frac{3}{2} + \frac{9xx}{4} - \frac{x}{2} + \frac{3}{2}x^3 + x - x^3 = 0$$

ſeu $1 - 6xx + 2x^3 + 9x^4 + 2x^5 = 0$. Cuius una radix eſt $x = -1$, cui reſpondet $y = 1$. At poſito y conſtante ſit $R = -6x - 2y$, ergo $\frac{ddy}{dx^2} = \frac{2y + 6x}{2y - xx}$; quod caſu $x = -1$ & $y = 1$ abit in -4, ita ut valor ipſius $y = 1$ ſit

ſit maximus. Ipſi $x = -1$ autem geminus valor ipſius y reſpondet ex aequatione $yy - y = 0$: alter ergo eſt $y = 0$, qui neque maximus eſt neque minimus. Quodſi aequatio illa quinti gradus per $x + 1$ dividatur, prodit aequatio, cuius radices ſimpliciter exhiberi nequeunt.

EXEMPLUM III.

Sit propoſita haec aequatio: $yy + 2xxy + 4x - 3 = 0$ *ex qua maximi minimive valores ipſius* y *requiruntur.*

Per differentiationem ergo prodibit haec aequatio:

$$2ydy + 2xxdy + 4xydx + 4dx = 0$$

Factoque $\frac{dy}{dx} = 0$ erit $xy + 1 = 0$, ideoque $y = -\frac{1}{x}$, quo valore ſubſtituto in ipſa aequatione propoſita oritur,

$$\frac{1}{xx} - 2x + 4x - 3 = 0 = 2x^3 - 3xx + 1$$

cuius radices ſunt $x = 1$; $x = 1$; & $x = -\frac{1}{2}$. Quia nunc eſt $\frac{dy}{dx} = \frac{-4xy - 4}{2y + 2xx} = \frac{-2xy - 2}{y + xx}$, erit differentiando $\frac{ddy}{dx^2} = -\frac{2y}{y + xx}$, poſito y conſtanti ob $dy = 0$ & facto $xy + 1 = 0$. Quare iſti valores ita ſe habebunt

x	y	$\frac{ddy}{dx^2}$	
1	-1	∞	
1	-1	∞	
$-\frac{1}{2}$	2	$-\frac{16}{9}$	pro maximo.

Quoniam pro radicibus aequalibus fit $\frac{ddy}{dx^2} = \infty$, utrum hoc caſu maximum an minimum prodeat non determinatur. Quia autem ſimul fit $y + xx = 0$; nequidem hoc caſu erit

 dy

$\frac{dy}{dx} = 0$; ob $P = 0$ & $Q = 0$ in fractione $\frac{dy}{dx} = -\frac{P}{Q}$; quare cum primaria proprietas deſit, neque maximum nec minimum habet locum. Indicatur autem hoc caſu $x = 1$, ambos ipſius y valores inter ſe fieri aequales. Quam indolem infra fuſius ſumus expoſituri, cum ad uſum calculi differentialis in doctrina de lineis curvis perveniemus. Etiamſi enim haec materia & huc pertineat; tamen ne eam bis attingere opus ſit, eam totam ſequenti tractationi reſervamus.

278. Datur vero inſuper in functionibus multiformibus alia ſpecies maximorum ac minimorum; quae methodo hactenus tradita non invenitur, cuius natura ex functionibus biformibus facillime explicari poteſt. Sit enim y functio quaecunque biformis ipſius x, ita ut, quicunque valor ipſi x tribuatur, pro y oriantur bini valores vel ambo reales vel ambo imaginarii. Ponamus hos ipſius y valores fieri imaginarios, ſi ponatur $x > f$, reales autem eſſe, ſi ſtatuatur $x < f$; atque poſito $x = f$ ambo ipſius y valores in unum coaleſcent, qui ſit $y = g$. Cum igitur ſi ſumatur $x > f$; functio y nullum habeat valorem realem: ſi eveniat, ut poſito $x < f$ ambo ipſius y valores fiant vel maiores quam g, vel minores quam g: priori caſu valor $y = g$ erit minimus, poſteriori maximus; quoniam illo caſu minor eſt, quam ambo praecedentes, hoc vero maior. Neque hoc maximum minimumve methodo hactenus tradita reperietur, propterea quod hic non ſit $\frac{dy}{dx} = 0$. Sunt autem quoque haec maxima vel minima generis diverſi, cum talia non ſint ratione valorum antecedentium & conſequentium in ſerie cohaerentium; ſed ratione binorum valorum disiunctorum vel antecedentium vel ſequentium tantum.

279. Evenit hoc ſi aequatio propoſita fuerit huiuſmodi $y = p \pm (f - x) \surd (f - x) q$, exiſtentibus p & q functionibus ipſius x per $f - x$ non diviſibilibus; obtineatque q valo-

valorem affirmativum, ſi ponatur vel $x = f$ vel aliquanto maius minuſve. Fiat $p = g$ poſito $x = f$: & manifeſtum eſt, caſu $x = f$ ambos ipſius y valores in unum $y = g$ coaleſcere; poſito autem $x > f$ ambo valores ipſius y fient imaginarii. Si igitur ponamus x aliquanto minus quam f, puta $x = f - \omega$, functio p abibit in $g - \frac{\omega d p}{d x} + \frac{\omega^2 d d p}{2 d x^2} -$ &c. & q in $q - \frac{\omega d q}{d x} + \frac{\omega^2 d d q}{2 d x^2} -$ &c. unde hoc caſu erit

$$y = g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2 dx^2} - \&c. \pm \omega \sqrt{\omega} \left(q - \frac{\omega d q}{dx} + \frac{\omega^2 ddq}{2dx^2} - \&c. \right).$$

Ponamus ω minimum, ut prae ω altiores eius poteſtates evaneſcant, eritque $y = g - \frac{\omega d p}{d x} \pm \omega \sqrt{\omega q}$; qui valores ambo ipſius y minores erunt quam g, ſi $\frac{dp}{dx}$ fuerit affirmativum, maiores autem, ſi negativum. Unde valor duplex ipſius $y = g$ illo caſu erit maximus, hoc vero minimus.

280. Haec igitur maxima atque minima inde ortum ſuum habent, quod primo poſito $x = f$ ambo ipſius y valores fiant aequales: poſito autem $x > f$ imaginarii, at poſito $x < f$ reales. Deinde quod poſito $x = f - \omega$ alterum membrum irrationale praebeat altiores poteſtates ipſius ω, quam membrum rationale. Hoc ergo evenit quoque ſi fuerit $y = p \pm (f-x)^n \sqrt{(f-x)} q$, dummodo ſit n numerus integer > 0. Cum autem non ſolum radix quadrata ſed etiam quaecunque alia radix poteſtatis paris eandem ambiguitatem ſignorum introducat; idem eveniet, ſi fuerit $y = p \pm (f-x)^{\frac{2n+1}{2m}} q$, dummodo ſit $2n + 1 > 2m$, erit ergo $(y-p)^{2m} = (f-x)^{2n+1} q^{2m}$ ſeu $(y-p)^{2m} = (f-x)^{2n+1} Q$. Quoties ergo functio y per huiuſmodi aequationem exprimitur, ita ut ſit $2n + 1 > 2m$, toties poſito $x = f$, valor ipſius y fiet maximus vel minimus:

mus: prius quidem ſi fuerit $\frac{dp}{dx}$ quantitas affirmativa, poſterius vero ſi ſit $\frac{dp}{dx}$ quantitas negativa poſito $x = f$. Sin autem fiat hoc caſu $\frac{dp}{dx} = 0$, tum erit $y = g + \frac{\omega^2 ddp}{2dx^2} \pm \omega^{\frac{2n+1}{2m}} q$. Niſi ergo ſit $\frac{2n+1}{2m} > 2$, neque maximum neque minimum locum habebit; at ſi $\frac{2n+1}{2m} > 2$, tum $y = g$ erit maximum, ſi $\frac{ddp}{dx^2}$ habuerit valorem negativum, minimum vero, ſi affirmativum: ſicque ulterius ſi quoque $\frac{ddp}{dx^2}$ evaneſcat, iudicium erit inſtituendum.

281. Si igitur y fuerit huiuſmodi functio ipſius x, fieri poteſt, ut praeter maxima & minima, quae prior methodus exhibet, etiam maxima minimave huius alterius ſpeciei adſint, quae modo hic expoſito explorari poterunt. Id quod ſequentibus exemplis declarabimus.

EXEMPLUM I.

Determinare maxima ac minima functionis y, *quae definitur hac aequatione:*

$$yy - 2xy - 2xx - 1 + 3x + x^3 = 0.$$

Ad maxima minimave primae ſpeciei inveſtiganda differentietur aequatio, eritque

$$2ydy - 2xdy - 2ydx - 4xdx + 3dx + 3xxdx = 0,$$

poſitoque $\frac{dy}{dx} = 0$, erit $y = \frac{3}{2} - 2x + \frac{3}{2}xx$,

qui valor in prima aequatione ſubſtitutus dat:

$9x^4 - 32x^3 + 42xx - 24x + 5 = 0$, quae reſolvitur in

$$9xx - 14x + 5 = 0 \quad \& \quad xx - 2x + 1 = 0.$$

Poſte-

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Poſterior bis dat $x=1$, ſitque $y=1$, unde hoc caſu in fractione $\frac{dy}{dx}=\frac{2y-3+4x-3xx}{2y-2x}$ denominator quoque evaneſcit, ſicque maximum minimumve primi generis non datur: prior vero aequatio $9xx-14x+5=0$ dabit $x=1$ & $x=\frac{5}{9}$, quorum valorum ille eodem incommodo laborat, quo praecedentes. Poſito autem $x=\frac{5}{9}$, fit $y=\frac{3}{2}-\frac{10}{9}+\frac{25}{54}=\frac{23}{27}$. Et cum ſit $\frac{dy}{dx}=\frac{2y-3+4x-3xx}{2y-2x}$, fiet $\frac{ddy}{dx^2}=\frac{+4-6x}{2y-2x}=\frac{-3x+2}{y-x}$ ob $dy=0$ & numeratorem $=0$. Erit ergo $\frac{ddy}{dx^2}=\frac{9}{8}$, unde hic valor $x=\frac{5}{9}$ dat minimum primi generis. Deinde cum ſit $(y-x)^2=(1-x)^3$, erit $y=x\pm(1-x)\surd(1-x)$; ideoque poſito $x=1$ prodit maximum ſecundae ſpeciei: facto enim $x=1-\omega$, erit $y=1-\omega\pm\omega\surd\omega$, quorum uterque minor eſt quam unitas, ſiquidem ω ſumatur minimum.

EXEMPLUM II.

Invenire maxima ac minima functionis:

$$y=2x-xx\pm(1-x)^2\surd(1-x).$$

Pro primi generis maximis & minimis differentietur aequatio, eritque:

$$\frac{dy}{dx}=2-2x\mp\frac{5}{2}(1-x)\surd(1-x)$$

quo valore poſito $=0$ prodit primo $x=1$ & cum ſit $\frac{ddy}{dx^2}=-2\pm\frac{15}{4}\surd(1-x)$, erit y hoc caſu maximum primi generis, ſitque $y=1$. Aequatione vero $\frac{dy}{dx}=0$ per

$1-x$

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$1 - x$ diviſa erit $4 \mp 5\sqrt{(1-x)} = 0$ ſeu $16 = 25 - 25x$, unde fit $x = \frac{9}{25}$, & $\frac{ddy}{dx^2} = -2 \pm 3$. Quare ſi ſignum ſuperius valet, erit $y = \frac{2869}{3125}$ minimum; ſin autem ſignum inferius valeat, erit $y = \frac{821}{3125}$, quod maximum videatur: at vero tantum ſignum ſuperius locum habere poteſt, quoniam $4 \mp 5\sqrt{(1-x)}$ nequit eſſe $= 0$, niſi ſit $\sqrt{(1-x)} = +\frac{4}{5}$. Primi ergo generis invenimus maximum caſu $x = 1$ & $y = 1$, atque minimum caſu $x = \frac{9}{25}$ & $y = \frac{2869}{3125}$. Ex genere vero altero maximum quoque prodit; ſi $x = 1$, quo caſu fit $y = 1$. Nam poſito $x = 1 - \omega$, erit $y = 1 - \omega\omega \pm \omega^2\sqrt{\omega}$ utroque caſu < 1. Hic itaque, ſi $x = 1$ maxima duo primae & alterius ſpeciei coaleſcunt, maximumque quaſi mixtum conſtituunt.

282. Ex his exemplis non ſolum natura huius alterius ſpeciei maximorum & minimorum elucet; ſed etiam pro lubitu iſtiuſmodi functiones formari poſſunt, quae maxima vel minima ſecundae ſpeciei admittant. Quemadmodum autem, ſi propoſita fuerit functio quaecunque, explorari poſſit, utrum eiuſmodi maximis minimiſve ſit praedita nec ne, id in ſequenti ſectione oſtendemus: propterea quod natura linearum curvarum hac inveſtigatione maxime illuſtratur. Ceterum vero facile intelligitur, ſi fuerit y eiuſmodi functio ipſius x, quae maximum minimumve ſecundae ſpeciei recipiat, tum quoque viciſſim x eiuſmodi fore functionem ipſius y. Nam quia ex hac aequatione $(y-x)^2 = (1-x)^3$, facto $x = 1$, obtinet y valorem maximum ſecundae ſpeciei; ſi variabiles y & x permutentur, haec aequatio $(x-y)^2 = (1-y)^3$ exhibet pro y quoque eiuſmodi functionem ipſius x, quae habeat maxi-

maximum ſecundae ſpeciei. Facto enim $x = 1$, fiet $(1-y)^2 = (1-y)^3$, hincque erit bis $y = 1$ & ſemel $y = 0$. Sin autem ponatur $x = 1 + \omega$; erit $(1+\omega-y)^2 = (1-y)^3$; unde ſi ſtatuamus $y = 1 + \varphi$ erit $(\omega-\varphi)^2 = (-\varphi)^3 = -\varphi^3$ ideoque φ debet eſſe negativum. Sit ergo $y = 1 - \varphi$ erit $(\omega+\varphi)^2 = \varphi^3$, atque cum ſumto φ minimo, φ^3 prae φ^2 evaneſcat, debebit neceſſario ω eſſe negativum: hinc valori $x = 1 + \omega$ nulli valores reales ipſius y reſpondent. At poſito $x = 1 - \omega$, & $y = 1 - \varphi$, ob $(\varphi-\omega)^2 = \varphi^3$, erit $\varphi = \omega \pm \omega\sqrt{\omega}$, ideoque $y = 1 - \omega \mp \omega\sqrt{\omega}$, unde uterque valor ipſius y reſpondens ipſi $x = 1 - \omega$ minor eſt valore $y = 1$, qui reſpondet valori $x = 1$; eritque conſequenter iſte ipſius y valor maximus.

283. Hactenus tantum functiones biformes ſumus contemplati, quarum maxima vel minima, quia ambo valores facile per reſolutionem aequationis quadraticae exprimi poſſunt, ad examen revocari poſſunt. Sin autem functio y per aequationem altiorem exprimatur, methodus ante tradita, qua maxima minimaque primae ſpeciei indagavimus, eodem ſucceſſu adhiberi poterit. Inventionem vero maximorum ac minimorum ſecundae ſpeciei ſequenti ſectioni reſervamus. Functiones ergo triformes ac multiformes, quemadmodum tractari oporteat, aliquot exemplis oſtendamus.

EXEMPLUM I.

Definiatur functio y, *cuius maxima vel minima quaeruntur, per hanc aequationem:*

$$y^3 + x^3 = 3axy.$$

Differentiata hac aequatione fit $3y^2dy + 3xxdx = 3axdy + 3aydx$, ideoque $\frac{dy}{dx} = \frac{ay - xx}{yy - ax}$. Maximum ergo vel minimum dabitur, ſi fuerit $ay = xx$, ſeu $y = \frac{xx}{a}$, qui valor in aequatione propoſita ſubſtitutus dat.

$$\frac{x^6}{a^3} + x^3 = 3x^3 \quad \text{ſeu} \quad x^6 = 2a^3x^3;$$

 Erit

Erit ergo ter $x = 0$, quo caſu quoque fit denominator $yy - ax = 0$, ob $y = \frac{xx}{a} = 0$. Utrum ergo hoc caſu maximum minimumve prodeat, patebit ſi ipſi x valorem tribuamus minime ab 0 diſcrepantem. Sit ergo $x = \omega$, & $y = \varphi$, ob $\varphi^3 + \omega^3 = 3a\omega\varphi$, fiet vel $\varphi = \alpha\sqrt{\omega}$ vel $\varphi = \beta\omega^2$. Priori caſu erit $\alpha^3\omega\sqrt{\omega} = 3\alpha a\omega\sqrt{\omega}$, ideoque $\alpha = \sqrt{3a}$. Hinc poſito $x = \omega$ erit $y = \pm\sqrt{3a\omega}$. Unde etiamſi ω negative accipi nequeat, tamen binorum ipſius y valorum alter maior erit quam 0, alter minor; hincque $y = 0$ neque maximum erit neque minimum. Sin autem ſtatuatur $\varphi = \beta\omega^2$ erit $\omega^3 = 3a\beta\omega^3$, ideoque $\beta = \frac{1}{3a}$ & $\varphi = \frac{\omega^2}{3a}$: Ergo hoc caſu ſive x capiatur $= +\omega$ ſive $= -\omega$, valor ipſius $y = \varphi$ nihilo erit maior, ideoque hoc caſu $y = 0$ erit minimum. Reſtat ergo tertius caſus ex aequatione $x^3 = 2a^3$ examinandus, qui dat $x = a\sqrt[3]{2}$, & $y = a\sqrt[3]{4}$. Qui utrum ſit maximus an minimus ex aequatione $\frac{dy}{dx} = \frac{ay - xx}{yy - ax}$ quaeratur differentiale ſecundum, quod ob $dy = 0$ & $ay - xx = 0$ erit $\frac{ddy}{dx^2} = \frac{-2x}{yy - ax}$, cuius valor praeſenti caſu eſt $-\frac{2a\sqrt[3]{2}}{2a^2\sqrt[3]{2} - aa\sqrt[3]{2}} = -\frac{2}{a}$, qui indicat valorem ipſius y eſſe maximum.

EXEMPLUM II.

Si functio y definiatur per hanc aequationem:

$$y^4 + x^4 + ay^3 + ax^3 = b^3x + b^3y,$$

invenire eius maximos minimoſve valores.

Cum

Cum per differentiationem oriatur
$4y^3dy + 3ayydy - b^3dy = b^3dx - 3axxdx - 4x^3dx$; erit
$\frac{dy}{dx} = \frac{b^3 - 3axx - 4x^3}{4y^3 + 3ayy - b^3}$, ponique oportet: $b^3 = 3axx + 4x^3$. Quaeſtio ergo huc reducitur, ut functionis uniformis $b^3x - ax^3 - x^4$ maxima ac minima indagentur, quae ſimul erunt maxima ſeu minima functionis y. Sit $a = 2$ & $b = 3$ ſeu proponatur haec aequatio $y^4 + x^4 + 2y^3 + 2x^3 = 27x + 27y$;
erit $\frac{dy}{dx} = \frac{27 - 6xx - 4x^3}{4y^3 + 6yy - 27}$ & $4x^3 + 6xx - 27 = 0$,
quae diviſa per $2x - 3 = 0$ dat $2xx + 6x + 9 = 0$, cuius poſterioris radices cum ſint imaginariae, erit $x = \frac{3}{2}$ &
$y^4 + 2y^3 - 27y = \frac{459}{16}$, cuius ſingulae radices erunt vel maximae vel minimae. Cum autem ſit
$\frac{dy}{dx} = \frac{27 - 6xx - 4x^3}{4y^3 + 6yy - 27}$, erit $\frac{ddy}{dx^2} = \frac{-12x - 12xx}{4y^3 + 6yy - 27}$,
qui poſito $x = \frac{3}{2}$, ſi fuerit affirmativus, indicabit minimum, contra vero maximum.

EXEMPLUM III.

Si fuerit $y^m + ax^n = by^px^q$; *definire maxima & minima ipſius* y.

Per differentiationem fit $\frac{dy}{dx} = \frac{qby^px^{q-1} - nax^{n-1}}{my^{m-1} - pby^{p-1}x^q}$, quo poſito $= 0$, erit primo $x = 0$, ſi quidem n & q fuerint unitate maiores; atque ſimul $y = 0$. Quo caſu an detur maximum vel minimum, valores proximi ſunt inveſtigandi, quoniam quoque denominator fit $= 0$; quae inveſtigatio ab exponentibus potiſſimum pendebit. Praeterea vero aequatio $\frac{dy}{dx} = 0$

 dabit

dabit $y^p = \frac{na}{qb} x^{n-q}$, qui valor in propoſita ſubſtitutus ponendo $\frac{na}{qb} = g$ dabit $g^{\frac{m}{p}} x^{\frac{mn-mq}{p}} + ax^n = \frac{na}{q} x^n$ ſeu $g^{\frac{m}{p}} x^{\frac{mn-mq-np}{p}} = \frac{(n-q)a}{q}$, unde fit $x = \left(\frac{(n-q)a}{q}\right)^{p:(mn-mq-np)} : g^{m:(mn-mq-np)}$ ſimulque valor ipſius y innoteſcit. Deinde diſpiciendum eſt, utrum differentio-differentiale

$$\frac{ddy}{dx^2} = \frac{q(q-1)by^p x^{q-2} - n(n-1)ax^{n-2}}{my^{m-1} - pby^{p-1}x^q}$$

obtineat valorem affirmativum an negativum, ut ex priori minimum, ex poſteriori vero maximum pronuncietur.

EXEMPLUM IV.

Si fuerit $y^4 + x^4 = 4xy - 2$, *maxima & minima functionis* y *aſſignare.*

Differentiatione inſtituta fit $\frac{dy}{dx} = \frac{y - x^3}{y^3 - x}$, hinque oritur $y = x^3$, erit ergo $x^{12} = 3x^4 - 2$ ſeu $x^{12} - 3x^4 + 2 = 0$, quae aequatio reſolvitur in has $x^4 - 1 = 0$ & $x^8 + x^4 - 2 = 0$, poſteriorque in $x^4 - 1 = 0$ & $x^4 + 2 = 0$. Hinc erit bis vel $x = +1$ vel $x = -1$; utroque vero caſu & denominator fractionis $\frac{dy}{dx}$ evaneſcit. Ad inveſtigandum ergo utrum his caſibus maximum minimumve locum habeat, ponamus $x = 1 - \omega$ & $y = 1 - \varphi$; erit:

$$\begin{array}{llll} 1 - 4\varphi & + 1 - 4\omega & = 4 - 4\omega - 4\varphi - 2 \\ + 6\varphi^2 + & 6\omega^2 & \quad + 4\omega \\ - 4\varphi^3 - & 4\omega^3 & \\ + \varphi^4 + & \varphi^4 & \end{array}$$

ideoque $4\omega\varphi = 6\varphi^2 + 6\omega^2 - 4\varphi^3 - 4\omega^3 + \varphi^4 + \omega^4$, & ob ω & φ minima $4\omega\varphi = 6\varphi^2 + 6\omega^2$. Valor ergo ipſius φ erit imaginarius, ſive ω capiatur affirmative ſive negative. Seu ſi y

& x

& x deſignent coordinatas curvae, ea caſu $x = 1$ & $y = 1$ habebit punctum coniugatum. Neque ergo hic valor pro maximo neque pro minimo haberi poteſt, propterea quod antecedentes & conſequentes, cum quibus comparari deberet, fiunt imaginarii.

284. Si aequatio, qua relatio inter x & y exprimitur, ita fuerit comparata, ut functio ipſius y aequetur functioni ipſius x, puta $Y = X$; ad maxima minimave invenienda poni debebit $dX = 0$: fiet ergo y maximum vel minimum iiſdem caſibus, quibus X fit maximum vel minimum. Simili modo ſi x tanquam functio ipſius y conſideretur, fiet x maximum vel minimum ſi $dY = 0$ hoc eſt ſi Y fuerit maximum vel minimum. Neque tamen hinc ſequitur y & x ſimul fieri maxima vel minima. Nam ſi fuerit $2ay - yy = 2bx - xx$, erit y maximum vel minimum, ſi fuerit $x = b$; eritque $y = a \pm \sqrt{(aa - bb)}$. Contra vero x fit maximum vel minimum, ſi fuerit $y = a$, fitque $x = b \pm \sqrt{(bb - aa)}$, neque ergo fiet y maximum vel minimum, ſi $x = b \pm \sqrt{(bb - aa)}$, quo tamen caſu x eſt maximum minimumve. Ceterum hoc caſu, ſi y habeat valores maximos vel minimos, x hac indole prorſus carebit: namque y maximum minimumve fieri nequit, niſi ſit $a > b$, quo caſu maximum minimumve ipſius x fit imaginarium.

285. Tum vero etiam evenire poteſt, ut non omnes radices aequationis $dX = 0$ praebeant maximos minimoſve valores pro y; ſi enim illa aequatio duas habuerit radices aequales, exinde neque maximum neque minimum conſequitur; hocque idem evenit, ſi quotcunque radices numero pares fuerint inter ſe aequales. Sic ſi proponatur aequatio $b(y - a)^2 = (x - b)^3 + c^3$; quia ſumtis differentialibus fit $2bdy(y - a) = 3dx(x - b)^2$, functio y neque maxima fiet neque minima poſito $x = b$, propterea quod hic occurrunt duae radices aequales. Sin autem x tanquam functio ipſius y ſpectetur, ea fiet maxima vel minima, ſi ſtatuatur $y = a$; eritque $x = b - c$ minimum. Quia denique in huiuſmodi aequa-

quationibus $Y = X$ variabiles x & y inter ſe non permiſcentur, ſi ipſi x tribuitur valor, qui ſit radix aequationis $dX = 0$, omnes valores ipſius y, quotcunque fuerint reales, erunt maximi vel minimi; quod non evenit, ſi in aequatione ambae variabiles fuerint permixtae.

286. Quae praeterea ſuperſunt de natura maximorum ac minimorum exponenda; ea in ſequentem ſectionem reſervamus, quoniam commodius ope figurarum menti repraeſentari atque explicari poſſunt. Pergamus ergo ad functiones, quae ex pluribus variabilibus ſunt compoſitae, atque inveſtigemus valores, quos ſingulis variabilibus tribui oportet, ut ipſa functio vel maximum vel minimum valorem obtineat. Ac primo quidem patet, ſi variabiles non fuerint inter ſe permixtae, ita ut functio propoſita ſit huiuſmodi $X + Y$, exiſtente X functione ipſius x, & Y ipſius y tantum, tum functionem propoſitam $X + Y$ fore maximum, ſi ſimul X & Y maximum evadat: minimumque, ſi ſimul X & Y fiat minimum. Ad maximum ergo inveniendum inquirantur valores ipſius x, quibus X fiat maximum, ſimilique modo valores ipſius y, quibus Y ſit maximum: hique valores pro x & y inventi efficient functionem $X + Y$ maximam, quod ſimiliter de minimo erit tenendum. Cavendum ergo eſt, ne duo valores ipſarum x & y diverſae naturae combinentur, quorum ille reddat X maximum, hic vero Y minimum, aut contra. Hoc enim ſi fieret, functio $X + Y$ neque maximum foret neque minimum. At huiuſmodi functio $X - Y$ fiet maxima, ſi X fuerit maximum ſimulque Y minimum; contra vero $X - Y$ fiet minimum, ſi X fuerit minimum & Y maximum. Sin autem utraque functio X & Y ſtatueretur vel maxima vel minima, earum differentia $X - Y$ neque foret maxima, neque minima; quae omnia ſunt ex natura maximorum ac minimorum ante expoſita clara & perſpicua.

287. Si ergo quaerantur maximi minimive valores functionis duarum variabilium; quaeſtio multo magis cautioni ob-

obnoxia eſt, quam ſi unica fuerit variabilis. Non ſolum enim pro utraque variabili caſus, quibus maximum minimumve producitur, diligenter ſunt diſtinguendi; ſed etiam ex his bini eiuſmodi ſunt coniungendi, ut functio propoſita fiat maximum vel minimum; id quod ex exemplis clarius patebit.

EXEMPLUM I.

Sit propoſita haec duarum variabilium x *&* y *functio*:

$$y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3xx - 3x$$

& quaerantur valores pro y *&* x *ſubſtituendi, ut haec functio maximum vel minimum obtineat valorem.*

Quoniam haec expreſſio in duas huiuſmodi partes $Y + X$ reſolvitur, quarum illa eſt functio ipſius y, haec vero ipſius x tantum; caſus quibus utraque fit maxima vel minima, inveſtigentur. Cum igitur ſit

$$Y = y^4 - 8y^3 + 18y^2 - 8y, \text{ erit } \frac{dY}{dy} = 4y^3 - 24y^2 + 36y - 8$$

qua expreſſione nihilo aequali poſita, fiet per 4 diviſa $y^3 - 6y^2 + 9y - 2 = 0$: cuius radices ſunt $y = 2$, & $y = 2 \pm \sqrt{3}$. Cum ergo ſit $\frac{ddY}{4dy^2} = 3yy - 12y + 9$; caſu $y = 2$, prodibit maximum. Pro reliquis binis radicibus $y = 2 \pm \sqrt{3}$, quae oriuntur ex aequatione $yy - 4y + 1 = 0$ fiet $\frac{ddY}{12dy^2} = yy - 4y + 3 = 2$, unde utraque dat minimum. Erit autem his caſibus ut ſequitur.

$y = 2$	$Y = 8$ maximum
$y = 2 - \sqrt{3}$	$Y = -1$ minimum
$y = 2 + \sqrt{3}$	$Y = -1$ minimum

Simili modo cum ſit $X = x^3 - 3xx - 3x$, erit $\frac{dX}{dx} = 3xx - 6x - 3$, unde oritur haec aequatio $xx = 2x + 1$ & $x = 1 \pm \sqrt{2}$. Eſt vero $\frac{ddX}{6dx^2} = x - 1 = \pm \sqrt{2}$. Ergo radix

dix $x = 1 + \sqrt{2}$ dat minimum, nempe $X = -5 - 4\sqrt{2}$, & $x = 1 - \sqrt{2}$ dat maximum, nempe $X = -5 + 4\sqrt{2}$. Quocirca formula propoſita $X + Y = y^4 - 8y^3 + 18yy - 8y + x^3 - 3xx - 3x$ fiet maxima, ſi ponatur $y = 2$ & $x = 1 - \sqrt{2}$, prodibitque $X + Y = 3 + 4\sqrt{2}$. Eadem autem formula $X + Y$ fiet minima, ſi ſumatur vel $y = 2 - \sqrt{3}$ vel $y = 2 + \sqrt{3}$ & $x = 1 + \sqrt{2}$, utroque caſu erit $X + Y = -6 - 4\sqrt{2}$.

EXEMPLUM II.

Si proponatur haec functio duarum variabilium:

$$y^4 - 8y^3 + 18y^2 - 8y - x^3 + 3xx + 3x$$

quae quibus caſibus fiat maxima vel minima inveſtigetur.

Poſito ut in praecedente exemplo habuimus, $Y = y^4 - 8y^3 + 18y^2 - 8y$ & $X = x^3 - 3xx - 3x$; formula propoſita erit $Y - X$; ideoque fiet maxima, ſi Y fuerit maximum & X minimum. Cum igitur hos caſus iam ante eruerimus, patet $Y - X$ obtinere valorem maximum, ſi ponatur $y = 2$ & $x = 1 + \sqrt{2}$; fietque $Y - X = 13 + 4\sqrt{2}$. Minimus vero valor ipſius $Y - X$ evadet, ſi Y ſit minimum, & X maximum, quod evenit ponendo $y = 2 \pm \sqrt{3}$ & $x = 1 - \sqrt{2}$, fiet autem $Y - X = 4 - 4\sqrt{2}$. Ceterum in utroque exemplo patet hos valores, quos invenimus, neque omnium eſſe maximos neque minimos: nam ſi utrinque poneretur verbi gratia $y = 100$ & $x = 0$, ſine dubio maior prodiret valor eo, quem invenimus: ſimilique modo ponendo $y = 0$ & vel $x = -100$ vel $x = +100$ minor prodiret valor, quam ſunt illi, quos pro caſu minimi invenimus. Probe ergo tenenda eſt idea ſupra expoſita, quam de natura maximorum ac minimorum dedimus. Scilicet eum valorem vocari maximum, qui maior ſit valoribus tam antecedentibus quam conſequentibus contiguis proximis; minimum autem eſſe eum, qui his valoribus tam antecedentibus quam conſequentibus fuerit minor. Sic in hoc exemplo eſt valor ipſius $Y - X$, qui prodit ponendo $y = 2$ & $x = 1 + \sqrt{2}$ maior eſt eo, qui reſul-

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tat ſi ponatur $y = 2 \pm \omega$ & $x = 1 + \sqrt{2} \pm \varphi$ ſumtis pro ω & φ quantitatibus ſatis exiguis.

288. His exemplis expeditis facilior erit via ad ſolutionem generalem indagandam. Denotet V functionem quamcunque duarum variabilium x & y, ſintque pro x & y valores inveniendi, qui functioni V inducant maximum vel minimum valorem. Cum igitur ad hoc efficiendum utrique variabili x & y determinatus valor tribui debeat; ponamus alteram y iam habere eum valorem, qui requiritur ad functionem V vel maximam vel minimam reddendam: hocque poſito tantum opus erit, ut pro altera x idoneus quoque valor inveſtigetur, quod fiet, dum functio V differentiatur ponenda ſola x variabili, differentialeque nihilo aequale ſtatuitur. Simili modo ſi fingamus variabilem x iam eum habere valorem, qui aptus ſit ad functionem V vel maximam vel minimam efficiendam, valor ipſius y reperietur differentiando V poſita ſola y variabili, hocque differentiale nihilo aequali ponendo. Hinc ſi differentiale functionis V fuerit $= Pdx + Qdy$, oportebit eſſe & $P = 0$ & $Q = 0$, ex quibus duabus aequationibus valores utriuſque variabilis x & y erui poterunt.

289. Quoniam vero hoc pacto ſine diſcrimine reperiuntur valores pro x & y, quibus functio V vel maxima vel minima redditur; caſus, quibus vel maximum vel minimum oritur, probe a ſe invicem ſunt diſtinguendi. Ut enim functio V fiat maxima, neceſſe eſt ut ambae variabiles ad hoc conſpirent; namque ſi altera maximum exhiberet, altera minimum, ipſa functio neque maxima neque minima evaderet. Quocirca inventis ex aequationibus $P = 0$ & $Q = 0$ valoribus ipſarum x & y inquirendum eſt, utrum ambo ſimul functioni V vel maximum vel minimum valorem inducant; atque tum demum, cum compertum fuerit utriuſque variabilis valorem hinc erutum pro maximo valere, affirmare poterimus functionem hoc caſu maximum valorem induere. Quod idem de minimo erit tenendum, ita ut functio V minimum valorem

adipiſci nequeat, niſi ſimul ambae variabiles x & y minimum producant. Hinc ergo omnes illi caſus reiici debebunt, quibus altera variabilis maximum, altera vero minimum indicare deprehendetur. Interdum vero etiam evenit, ut alterius vel etiam utriuſque variabilis valores ex aequationibus $P = o$ & $Q = o$ oriundi neque maximum neque minimum exhibeant, qui caſus proinde pariter tanquam prorſus inepti erunt reiiciendi.

290. Utrum autem valores pro x & y reperti valeant pro maximo an minimo, de utroque ſeorſim ſimili modo inveſtigabitur, quo ſupra, cum unica adeſſet variabilis, ſumus uſi. Ad iudicium ſcilicet de variabili x inſtituendum conſideretur altera y tanquam conſtans, & cum ſit $dV = Pdx$ ſeu $\frac{dV}{dx} = P$, differentietur P denuo poſito y conſtante, ut prodeat $\frac{ddV}{dx^2} = \frac{dP}{dx}$, ac diſpiciatur, utrum valor ipſius $\frac{dP}{dx}$, poſtquam loco x & y valores ante inventi fuerint ſubſtituti, fiat affirmativus an negativus; priori enim caſu indicabitur minimum poſteriori vero maximum. Simili modo cum poſito x conſtante ſit $dV = Qdy$ ſeu $\frac{dV}{dy} = Q$, differentietur Q denuo poſita ſola y variabili, & examinetur valor $\frac{dQ}{dy}$, ſubſtitutis loco x & y valoribus, qui ex aequationibus $P = o$ & $Q = o$ ſunt inventi; qui ſi fuerit affirmativus, declarabit minimum, contra vero maximum. Hinc ergo colligitur, ſi ex valoribus pro x & y inventis formulae $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ induant valores diverſis ſignis affectos altera ſcilicet affirmativum, altera negativum, tum functionem V neque maximam neque minimam effici; ſin autem utraque formula $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ fiat affirmativa, minimum reſultabit; contraque, ſi utraque fiat negativa, maximum.

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291. Quodſi vero altera formula $\frac{dP}{dx}$ & $\frac{dQ}{dy}$, vel etiam utraque, ſi pro x & y valores inventi ſubſtituantur, evaneſcat, tum progrediendum erit ad differentialia ſequentia $\frac{ddP}{dx^2}$ & $\frac{ddQ}{dy^2}$, quae niſi pariter evaneſcant, neque maximum neque minimum habebit locum; ſin autem evaneſcant, iudicium ex formulis differentialibus ſequentibus $\frac{d^3P}{dx^3}$ & $\frac{d^3Q}{dy^3}$ erit petendum, ſimilique modo inſtituendum, quo pro formulis $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ eſt factum. Quo autem, quibus caſibus hoc uſu veniat, clarius exponamus, prodierit valor $x = \alpha$, qui ſi formulam $\frac{dP}{dx}$ reddat evaneſcentem, neceſſe eſt ut $\frac{dP}{dx}$ factorem habeat $x - \alpha$; qui factor ſi fuerit ſolitarius, neque ſimul alium ſibi habeat aequalem ſocium, neque maximum neque minimum indicabitur, quod idem evenit ſi $\frac{dP}{dx}$ factorem habuerit $(x-\alpha)^3$, vel $(x-\alpha)^5$, &c. Sin autem factor fuerit $(x-\alpha)^2$, vel $(x-\alpha)^4$, &c. tum quidem maximum vel minimum indicabitur; at inſuper videndum erit, utrum cum caſu per y indicato conſentiat.

292. Labor autem his caſibus ad differentialia ulteriora progrediendi mirifice ſublevari poterit: ſi enim ponamus, ut rem generalius complectamur, inventum eſſe $\alpha x + 6 = 0$, atque formulam $\frac{dP}{dx}$ factorem habere $(\alpha x + 6)^2$, ita ut ſit $\frac{dP}{dx} = (\alpha x + 6)^2\, T$, quia eſt $\alpha x + 6 = 0$, fiet $\frac{d^3P}{dx^3} = 2\alpha^2 T$, hincque ob $2\alpha^2$ affirmativum, ex ipſa quantitate T iudicium abſolvi poterit; quae ſi induet valorem affirmativum, pro minimo, contra vero pro maximo pronunciabit. Hocque idem

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ſubſidium in maximorum minimorumque inveſtigatione, ſi unica inſit variabilis adhiberi poterit, ita ut nunquam opus ſit ad altiora differentialia aſcendere. Quin etiam nequidem ad differentialia ſecunda procedere opus erit: ſi enim ex aequatione $P = 0$, fiat $\alpha x + \beta = 0$, neceſſe eſt ut P factorem habeat $\alpha x + \beta$; ſit $P = (\alpha x + \beta)T$, & cum ſit $\frac{dP}{dx} = \alpha T + (\alpha x + \beta)\frac{dT}{dx}$, ob $\alpha x + \beta = 0$, erit $\frac{dP}{dx} = \alpha T$, hincque iam ipſe alter factor T, prout valor ipſius αT fuerit vel affirmativus vel negativus, ſtatim vel minimum vel maximum indicabit.

293. His igitur traditis praeceptis haud difficile erit, ſi functio quaecunque duas variabiles involvens fuerit propoſita, caſus inveſtigare, quibus haec functio fiat vel maxima vel minima. Si quae inſuper notanda fuerint, ea ipſa exemplorum evolutio ſuggeret, quamobrem aliquot exemplis regulas datas illuſtrari expediet.

EXEMPLUM I.

Sit propoſita iſta functio duarum variabilium $V = xx + xy + yy - ax - by$, *quae quibus caſibus fiat vel maxima vel minima inquiratur.*

Cum ſit $dV = 2xdx + ydx + xdy + 2ydy - adx - bdy$, ſi comparetur cum formula generali $dV = Pdx + Qdy$ erit $P = 2x + y - a$ & $Q = 2y + x - b$: unde formabuntur iſtae aequationes $2x + y - a = 0$ & $2y + x - b = 0$, quibus coniunctis eliminando y fiet $x - b = 4x - 2a$, ideoque $x = \frac{2a - b}{3}$, & $y = a - 2x = \frac{2b - a}{3}$. Cum igitur ſit $\frac{dP}{dx} = 2$ & $\frac{dQ}{dy} = 2$, utraque oſtendit minimum; ex quo concludimus formulam $xx + xy + yy - ax - by$ fieri minimam, ſi ponatur

ponatur $x = \frac{2a - b}{3}$ & $y = \frac{2b - a}{3}$, prodibitque hoc modo $V = \frac{-3aa + 3ab - 3bb}{9} = \frac{-aa + ab - bb}{3}$, qui cum ſit unicus, omnium erit minimus. Unico ergo modo fieri poteſt $xx + xy + yy - ax - by = \frac{-aa + ab - bb}{3}$, & quia minor fieri nequit, erit haec aequatio $xx + xy + yy - ax - by = \frac{-aa + ab - bb}{3} - cc$ impoſſibilis.

EXEMPLUM II.

Si proponatur formula $V = x^3 + y^3 - 3axy$, *quaerantur caſus, quibus* V *adipiſcatur valorem maximum vel minimum.*

Ob $dV = 3xxdx + 3yydy - 3aydx - 3axdy$ erit $P = 3xx - 3ay$ & $Q = 3yy - 3ax$, unde fit $ay = xx$ & $ax = yy$. Cum ergo ſit $yy = x^4 : aa = ax$ erit $x^4 - a^3x = 0$; ideoque vel $x = 0$ vel $x = a$. Priori caſu fit $y = 0$, poſteriori vero $y = a$. Quoniam ergo eſt $\frac{dP}{dx} = 6x$, $\frac{ddP}{dx^2} = 6$, & $\frac{dQ}{dy} = 6y$ atque $\frac{ddQ}{dy^2} = 6$; priori ergo caſu, quo $x = 0$ & $y = 0$, neque maximum neque minimum reſultat. Poſteriori vero caſu quo & $x = a$ & $y = a$ minimum prodit, ſi quidem a fuerit quantitas affirmativa, fietque $V = -a^3$, qui autem valor tantum minor eſt proximis antecedentibus & conſequentibus: nam ſine dubio V multo minorem induere poteſt valorem, ſi utrique variabili x & y valores negativi tribuantur.

EXEMPLUM III.

Propoſita ſit haec functio $V = x^3 + ayy - bxy + cx$, *cuius valores maximi ſeu minimi inquirantur.*

Quia eſt $dV = 3xxdx + 2aydy - bydx - bxdy + cdx$ erit

P =

$P = 3xx - by + c$ & $Q = 2ay - bx$, quibus valoribus nihilo aequalibus poſitis erit $y = \frac{bx}{2a}$, ideoque $3xx - \frac{bbx}{2a} + c = 0$ ſeu $xx = \frac{2bbx - 4ac}{12a}$ unde fit $x = \frac{bb \pm \sqrt{(b^4 - 48aac)}}{12a}$.

Niſi ergo ſit $b^4 - 48aac > 0$, neque maximum neque minimum habet locum. Ponamus ergo eſſe $b^4 - 48aac = bbff$, ut ſit $c = \frac{bb(bb-ff)}{48aa}$; erit $x = \frac{bb \pm bf}{12a}$ & $y = \frac{bb(b \pm f)}{24aa}$. Quoniam porro eſt $\frac{dP}{dx} = 6x$ & $\frac{dQ}{dy} = 2a$, fiet $\frac{dP}{dx} = \frac{b(b \pm f)}{2a}$. Niſi ergo $2a$ & $\frac{b(b \pm f)}{2a}$ ſint quantitates eiuſdem ſigni, neque maximum neque minimum habet locum. At ſi ſint ambae vel affirmativae vel ambae negativae, quod evenit, ſi earum productum $b(b \pm f)$ fuerit affirmativum; tum functio V evadet minimum, ſi a ſit quantitas affirmativa, contra vero maximum, ſi a ſit quantitas negativa. Hinc ſi fuerit $f = 0$ ſeu $c = \frac{b^4}{48aa}$, ob bb quantitatem affirmativam, functio V evadet minima ſi a ſit quantitas poſitiva, ponaturque $x = \frac{bb}{12a}$ & $y = \frac{b^3}{24aa}$; contra vero ſi a ſit negativum, iſtae ſubſtitutiones producent maximum. Si ſit $f < b$, duobus caſibus oritur vel maximum vel minimum: at ſi $f > b$, tum caſus tantum $x = \frac{b(b+f)}{12a}$ & $y = \frac{bb(b+f)}{24aa}$ praebebit maximum minimumve prout a fuerit vel negativum vel affirmativum. Sit $a = 1$, $b = 3$ & $f = 1$, ut habeatur haec formula $V = x^3 + yy - 3xy + \frac{1}{2}x$ haec fiet minima ob a affirmativum, ſi ponatur vel $x = 1$ & $y = \frac{3}{2}$ vel $x = \frac{1}{2}$ & $y = \frac{3}{4}$. Priori

ri caſu oritur $V = \frac{1}{4}$, poſteriori vero $V = \frac{6}{16}$. Interim tamen patet loco x numeris negativis ponendis multo minores valores pro V oriri poſſe. Ita ergo intelligi debet valor ipſius $V = \frac{1}{4}$ minor eſſe, quam ſi ponatur $x = 2 + \omega$ & $y = 3 + \varphi$; dummodo ſint ω & φ numeri parvi, ſive affirmativi ſive negativi; limes autem quem ω tranſgredi non debet eſt $-\frac{15}{4}$; nam ſi $\omega < -\frac{15}{4}$ fieri poterit ut V fiat minor quam $\frac{1}{4}$.

EXEMPLUM IV.

Invenire maxima vel minima huius functionis:

$$V = x^4 + y^4 - axxy - axyy + ccxx + ccyy.$$

Sumto differentiali erit $P = 4x^3 - 2axy - ayy + 2ccx$ & $Q = 4y^3 - axx - 2axy + 2ccy$, quibus valoribus nihilo aequalibus poſitis, ſi a ſe invicem ſubtrahantur erit: $4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0$, quae cum ſit diviſibilis per $x - y$, erit primo $y = x$, atque $4x^3 - 3axx + 2ccx = 0$, quae dat $x = 0$ & $4xx = 3ax - 2cc$ ſeu $x = \frac{3a \pm \sqrt{(9aa - 32cc)}}{8}$.

Si ſumamus $x = 0$, erit quoque $y = 0$; & ob $\frac{dP}{dx} = 12xx - 2ay + 2cc$, atque $\frac{dQ}{dy} = 12yy - 2ax + 2cc$, fiet functio V minima $= 0$. Sin ſtatuamus $x = y = \frac{3a \pm \sqrt{(9aa - 32cc)}}{8}$, ſi quidem fuerit $9aa > 32cc$, ob $4xx = 3ax - 2cc$, erit $\frac{dP}{dx} = \frac{dQ}{dy} = 12xx - 2ax + 2cc = 7ax - 4cc = \frac{21aa - 32cc \pm 7a\sqrt{(9aa - 32cc)}}{8}$, qui valor cum ſit ſemper affirmativus ob $32cc < 9aa$, valor V hoc quoque caſu fit minimus, eritque $V = \frac{-27}{256}a^4 + \frac{9}{16}aacc - \frac{1}{2}c^4 \mp \frac{a}{256}(9aa - 32cc)^{\frac{3}{2}}$. Dividamus autem aequationem $4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0$ per $x - y$ fietque $4xx + 4xy + 4yy + ax + ay + 2cc = 0$. At ex aequatione $P = 0$, erit $yy = -2xy + \frac{4}{a}x^3 + \frac{2ccx}{a}$,

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quo valore in illa ſubſtituto fit

$$y = \frac{16x^3 + 4axx + aax + 8ccx + 2acc}{4ax - aa}.$$

Verum illa dat $y = -x \pm \sqrt{\frac{4x^3 + axx + 2ccx}{a}}$, unde efficitur:

$$16x^3 + 8axx + 4ccx + 2acc = (4x - a)\sqrt{(4ax^3 + aaxx + 2accx)},$$

quae ad rationalitatem perducta dat

$$\begin{array}{l} 256x^6 + 192ax^5 + 80aa\,x^4 + 4a^3\,x^3 - a^4\,x^2 - 2a^3cc\,x + 4a^2c^4 = 0 \\ \qquad\qquad\qquad\qquad + 128cc \quad + 96acc \quad + 48aacc \quad + 16ac^4 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + 16c^4 \end{array}$$

cuius radices, ſi quas habet, reales indicabunt maxima vel minima functionis V, ſi quidem $\frac{dP}{dx}$ & $\frac{dQ}{dy}$ fiant quantitates eodem ſigno affectae.

EXEMPLUM V.

Invenire maxima & minima huius expreſſionis:

$$x^4 + mxxyy + y^4 + aaxx + naaxy + aayy = V.$$

Facta differentiatione erit:

$$P = 4x^3 + 2mxyy + 2aax + naay = 0$$

$$Q = 4y^3 + 2mxxy + 2aay + naax = 0$$

quae aequationes invicem vel ſubtractae vel additae dant:

$$(4xx + 4xy + 4yy - 2mxy + 2aa - naa)(x - y) = 0$$

$$(4xx - 4xy + 4yy + 2mxy + 2aa + naa)(x + y) = 0$$

quae diviſae per $x - y$ & $x + y$, & denuo vel additae vel ſubtractae dant:

$$4xx + 4yy + 2aa = 0 \text{ \& } 4xy - 2mxy - naa = 0.$$

Ex quarum poſteriori fit $y = \frac{naa}{2(2-m)x}$, prior autem reales valores non admittit. Tres igitur habemus caſus:

I. Sit $y = x$, eritque $4x^3 + 2mx^3 + 2aax + naax = 0$, unde fit vel $x = 0$ vel $2(2 + m)xx + (2 + n)aa = 0$. Sit $x = 0$, erit quoque $y = 0$, atque ob $\frac{dP}{dx} = 12xx + 2myy + 2aa$

&

& $\frac{dQ}{dy} = 12yy + 2mxx + 2aa$, hoc caſu fiet $V = o$ minimum, ſi quidem coefficiens aa fuerit affirmativus. Alter caſus dat $xx = -\frac{(n+2)\,aa}{2\,(m+2)}$, quae realis eſſe nequit niſi ſit $\frac{n+2}{m+2}$ numerus negativus. Sit $\frac{n+2}{m+2} = -2kk$, ſeu $n = -2kkm - 4kk - 2$, erit $x = \pm ka$ & $y = \pm ka$. At $\frac{dP}{dx} = 12kkaa + 2mkkaa + 2aa$ & $\frac{dQ}{dy} = 12kkaa + 2mkkaa + 2aa$, quae cum ſint aequales, erit V vel minimum vel maximum, prout iſtae quantitates fuerint vel affirmativae vel negativae.

II. Sit $y = -x$, eritque $2(m+2)x^3 = (n-2)\,aax$ ergo vel $x = o$ vel $xx = \frac{(n-2)\,aa}{2\,(m+2)}$. Prior radix $x = o$ recidit in praecedentem. Poſterior vero erit realis ſi $\frac{(n-2)\,aa}{2(m+2)}$ fuerit quantitas affirmativa: & cum ſiat $\frac{dP}{dx} = \frac{dQ}{dy}$ prodibit vel maximum vel minimum.

III. Sit $y = \frac{naa}{2(2-m)x}$, erit $4x^3 + \frac{mn^2a^4}{2(2-m)^2x} + 2aax + \frac{nna^4}{2(2-m)x} = o$ ſeu $4x^4 + 2aaxx + \frac{nna^4}{(2-m)^2} = o$, cuius aequationis nulla radix eſt realis, niſi ſit aa quantitas negativa.

EXEMPLUM VI.

Propoſita ſit haec functio determinata:

$$V = x^4 + y^4 - xx + xy - yy,$$

cuius valores maximi vel minimi inveſtigentur.

Cum hinc fiat $P = 4x^3 - 2x + y = o$ & $Q = 4y^3 - 2y + x = o$ erit ex priori $y = 2x - 4x^3$, qui in altera ſubſtitutus dat $256x^9 - 384x^7 + 192x^5 - 40x^3 + 3x = o$. Cuius una radix

dix eſt $x=0$, unde fit quoque $y=0$. Ergo hoc caſu ob $\frac{dP}{dx}=12xx-2$ & $\frac{dQ}{dy}=12yy-2$ prodit maximum $V=0$. Diviſa autem aequatione inventa per x erit:

$$256x^8-384x^6+192x^4-40xx+3=0,$$

quae factorem habet $4xx-1$, unde fit $4xx=1$ & $x=\pm\frac{1}{2}$; atque $y=\pm\frac{1}{2}$, tum vero erit $\frac{dP}{dx}=\frac{dQ}{dy}=1$, utroque ergo caſu oritur minimum $V=-\frac{1}{8}$.

Dividatur illa aequatio per $4xx-1$, atque obtinebitur:

$$64x^6-80x^4+28xx-3=0,$$

quae denuo bis continet $4xx-1=0$; ita ut praecedens caſus oriatur. Praeterea vero inde fit $4xx-3=0$, & $x=\frac{\pm\sqrt{3}}{2}$; cui reſpondet $y=\frac{\mp\sqrt{3}}{2}$. Erit igitur quoque $\frac{dP}{dx}=\frac{dQ}{dy}=7$, ideoque fiet V minimum $=-\frac{9}{8}$; qui eſt valor omnium minimus, quos quidem functio V recipere poteſt: & hanc ob rem iſta aequatio $V=-\frac{9}{8}-cc$ ſemper eſt impoſſibilis. Hinc autem patet via determinandi maxima & minima functionum, quae tres pluresve variabiles involvunt.

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CAPUT XII.

DE USU DIFFERENTIALIUM IN INVESTIGANDIS RADICIBUS REALIBUS AEQUATIONUM.

294.

Natura maximorum ac minimorum viam nobis patefacit ad indolem radicum aequationum, utrum ſint reales an imaginariae, cognoſcendam. Sit enim propoſita aequatio cuiuſcunque ordinis:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$$

cuius radices ponamus eſſe p, q, r, s, t, &c. ita ut p ſit minima, q ea quae ratione magnitudinis ſequitur, ſicque & reliquae radices ſecundum ordinem quantitatis ſint diſpoſitae: ſcilicet ſit $q > p$; $r > q$; $s > r$; $t > s$ &c. Aſſumamus autem omnes radices aequationis eſſe reales, eritque exponens maximus n ſimul numerus radicum p, q, r, &c. Conſideremus quoque has radices omnes tanquam inter ſe inaequales; hinc tamen aequales radices non excluduntur, propterea quod radices inaequales, ſi earum differentia abeat in infinite parvam, fiant aequales.

295. Quoniam propoſita expreſſio $x^n - Ax^{n-1} + \&c.$ tum ſolum fit nihilo aequalis, cum loco x aliquis valor ex p, q, r, &c. ſubſtituitur, reliquis vero caſibus omnibus non evaneſcit, ponamus generatim:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = z$$

ita ut z ſpectari poſſit tanquam functio ipſius x. Fingamus nunc pro x ſucceſſive ſubſtitui valores determinatos, incipiendo a minimo $x = -\infty$, atque continuo maiores in locum ipſius x collocari; perſpicuumque eſt z nacturum hinc eſſe valores vel nihilo maiores vel nihilo minores, neque prius eſſe evaniturum, quam ponatur $x = p$; quo caſu fiet $z = 0$.

 Auge-

Augeantur valores ipſius x ultra p, atque valores ipſius z vel affirmativi vel negativi fient, donec perveniatur ad valorem $x = q$; quo caſu iterum erit $z = o$. Neceſſe ergo eſt, ut cum valores ipſius z ab o iterum ad o acceſſerint, interea z habuerit valorem vel maximum vel minimum; maximum ſcilicet ſi valores ipſius z, dum x intra limites p & q verſabatur, fuerint affirmativi, minimum, ſi fuerint negativi. Simili modo dum x ultra q ad r uſque augetur, functio z maximum vel minimum attinget: maximum nimirum ſi ante fuerit minimum, & contra. Supra enim vidimus maxima & minima ſe mutuo alternatim excipere.

296. Quare cum inter binas quaſvis radices ipſius x exiſtat caſus, quo functio z fit maximum vel minimum; erit numerus maximorum & minimorum, quae in functione z implicantur, unitate minor, quam numerus radicum realium; atque ita quidem alternatim ſe excipient, ut maximi ipſius z valores ſint affirmativi, minimi negativi. Quod ſi viciſſim functio z habeat maximum vel ſaltem valorem affirmativum caſu $x = f$, atque minimum ſeu ſaltem negativum caſu $x = g$; quoniam dum valores ipſius x ab f ad g tranſeunt, functio z ab affirmativo abit in negativum, neceſſe eſt ut interea per o tranſierit, & hancobrem dabitur radix ipſius x intra limites f & g contenta. Niſi autem haec conditio adſit, ut valores maximi minimique ipſius z fiant alternatim affirmativi & negativi, illa concluſio non ſequitur. Si enim dentur functionis z minima, quae quoque ſint affirmativa, fieri poteſt ut valor ipſius z a maximo ad ſequens minimum tranſeat, cum tamen interea non evaneſcat. Ceterum ex dictis intelligitur, etiamſi aequationis propoſitae non omnes radices fuerint reales, tamen ſemper inter binas quaſque dari maximum vel minimum; etiamſi propoſitio converſa generatim non valeat, ut inter bina quaevis maxima ſeu minima radix realis contineatur: valet autem adiecta conditione, ſi alter valor ipſius z fuerit affirmativus, alter negativus.

297. Quoniam ergo ſupra vidimus, valores ipſius x, quibus functio:

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c.$$

fit maximum vel minimum, eſſe radices aequationis differentialis huius:

$$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \&c. = 0$$

manifeſtum eſt, ſi aequationis $z = 0$ omnes radices, quarum numerus eſt $= n$ fuerint reales, tum quoque omnes radices aequationis $\frac{dz}{dx} = 0$ fore reales. Cum enim functio z tot habeat maxima vel minima, quot numerus $n - 1$ continet unitates, neceſſe eſt ut aequatio $\frac{dz}{dx} = 0$ totidem habeat radices reales; ideoque omnes eius radices erunt reales. Ex quo ſimul perſpicitur, functionem z plura maxima minimave habere non poſſe, quam $n - 1$. Habemus ergo hanc regulam latiſſime patentem, ſi aequationis $z = 0$ omnes radices fuerint reales, tum quoque aequatio $\frac{dz}{dx} = 0$ omnes radices habebit reales. Unde viciſſim ſequitur, ſi aequationis $\frac{dz}{dx} = 0$ non omnes radices fuerint reales, tum quoque non omnes aequationis $z = 0$ radices reales fore.

298. Quia inter binas quaſvis aequationis $z = 0$ radices reales datur unus caſus, quo functio z fit maximum vel minimum; ſequitur ſi aequatio $z = 0$ duas habeat radices reales, tum aequationem $\frac{dz}{dx} = 0$ neceſſario unam radicem habituram eſſe realem. Pariter ſi aequatio $z = 0$ tres habeat radices reales, tum aequatio $\frac{dz}{dx} = 0$ certo duas habebit radices rea-

reales. Atque generatim ſi aequatio $z = 0$ habeat m radices reales, neceſſe eſt ut aequationis $\frac{dz}{dx} = 0$ ad minimum ſint $m - 1$ radices reales. Quare ſi aequatio $\frac{dz}{dx} = 0$ pauciores habeat radices reales quam $m - 1$, tum viciſſim aequatio $z = 0$ certo pauciores quam m habebit radices reales. Cavendum autem eſt, ne propoſitio converſa pro vera habeatur; etiamſi enim aequatio differentialis $\frac{dz}{dx} = 0$ aliquot vel adeo omnes radices ſuas habeat reales, tamen non ſequitur, aequationem $z = 0$ ullam habituram eſſe radicem realem. Fieri enim poteſt, ut aequationis $\frac{dz}{dx} = 0$ omnes radices ſint reales, cum tamen aequationis $z = 0$ omnes radices ſint imaginariae.

299. Interim tamen, ſi conditio ſupra memorata adiiciatur, propoſitio converſa ita proponi poterit, ut ex radicibus realibus aequationis $\frac{dz}{dx} = 0$, numerus radicum realium aequationis $z = 0$ certo cognoſci poſſit. Ponamus enim $\alpha, \beta, \gamma, \delta$, &c. eſſe radices reales aequationis $\frac{dz}{dx} = 0$, inter quas α ſit maxima, reliquae vero ordine magnitudinis ſe invicem ſequantur. His igitur valoribus loco x ſubſtitutis functio z obtinebit vel maximos vel minimos valores alternatim. Cum autem functio z fiat $= \infty$, ſi ponatur $x = \infty$, patet eius valores continuo decreſcere debere, dum valores ipſius x ab ∞ uſque ad α diminuuntur; ex quo, caſu $x = \alpha$, fiet z minimum. Quodſi ergo hoc caſu $x = \alpha$ functio z valorem induat negativum, neceſſe eſt ut ante alicubi fuerit $= 0$, ſicque aequationis $z = 0$ radix dabitur realis $x > \alpha$; ſin autem poſito $x = \alpha$ functio z adhuc retineat valorem affirmativum,

tivum, ante nuſquam potuit eſſe minor, alias enim quoque daretur minimum antequam x ad α uſque diminueretur, quod eſſet contra hypotheſin; hinc aequatio $z = 0$ nullam habere poterit radicem realem maiorem quam α. Si ergo ponamus poſito $x = \alpha$ fieri $z = \mathfrak{A}$, hoc modo iudicari poterit: ſi fuerit $\mathfrak{A}$ quantitas affirmativa, tum aequatio $z = 0$ nullam habebit radicem realem α maiorem; ſin autem $\mathfrak{A}$ fuerit quantitas negativa, tum aequatio $z = 0$ unam perpetuo habebit radicem realem α maiorem, neque plures.

300. Ad hoc iudicium ulterius perſequendum

ſi ponatur	fiat
$x = \alpha$	$z = \mathfrak{A}$
$x = \beta$	$z = \mathfrak{B}$
$x = \gamma$	$z = \mathfrak{C}$
$x = \delta$	$z = \mathfrak{D}$
$x = \varepsilon$	$z = \mathfrak{E}$
&c.	&c.

Quia ergo $\mathfrak{A}$ fuit minimum, erit $\mathfrak{B}$ maximum, & quidem ſi $\mathfrak{A}$ fuerit affirmativum, erit quoque $\mathfrak{B}$ affirmativum, neque ergo inter limites α & β dabitur radix realis aequationis $z = 0$. Quare ſi haec aequatio nullam habeat radicem realem α maiorem, neque ullam habebit; quae eſſet maior quam β. Sin autem $\mathfrak{A}$ fuerit quantitas negativa, quo caſu una datur aequationis radix $x > \alpha$; diſpiciatur utrum valor ipſius $\mathfrak{B}$ ſit affirmativus an negativus, priori caſu dabitur radix $x > \beta$, poſteriori vero nulla dabitur radix intra limites α & β contenta. Simili modo cum $\mathfrak{B}$ fuerit maximum, erit $\mathfrak{C}$ minimum; quare ſi $\mathfrak{B}$ habuerit valorem negativum, multo magis $\mathfrak{C}$ erit negativum, nullaque hoc caſu dabitur radix intra limites β & γ contenta. At ſi $\mathfrak{B}$ fuerit affirmativum, radix dabitur realis inter limites β & γ, ſi $\mathfrak{C}$ fiat negativum: ſin autem $\mathfrak{C}$ quoque ſit affirmativum tum nulla dabitur radix inter limites β & γ contenta ſimilique modo iudicium ulterius erit inſtituendum.

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301. Quo haec iudicia facilius intelligantur, ea in ſequenti tabella complexus ſum:

Aequatio $z = 0$ unam habebit radicem realem, quae continetur intra limites	Si fuerit
$x = \infty$ & $x = \alpha$	$\mathfrak{A} = -$
$x = \alpha$ & $x = \beta$	$\mathfrak{A} = -$ & $\mathfrak{B} = +$
$x = \beta$ & $x = \gamma$	$\mathfrak{B} = +$ & $\mathfrak{C} = -$
$x = \gamma$ & $x = \delta$	$\mathfrak{C} = -$ & $\mathfrak{D} = +$
$x = \delta$ & $x = \varepsilon$	$\mathfrak{D} = +$ & $\mathfrak{E} = -$
&c.	&c.

Harumque propoſitionum converſae & in negantes tranſmutatae pariter in omni rigore locum obtinent. Scilicet

Aequatio $z = 0$ nullam habebit radicem realem, quae contineatur inter limites:	Si non fuerit
$x = \infty$ & $x = \alpha$	$\mathfrak{A} = -$
$x = \alpha$ & $x = \beta$	$\mathfrak{A} = -$ & $\mathfrak{B} = +$
$x = \beta$ & $x = \gamma$	$\mathfrak{B} = +$ & $\mathfrak{C} = -$
$x = \gamma$ & $x = \delta$	$\mathfrak{C} = -$ & $\mathfrak{D} = +$
$x = \delta$ & $x = \varepsilon$	$\mathfrak{D} = +$ & $\mathfrak{E} = -$
&c.	&c.

Ope harum ergo regularum ex radicibus aequationis $\frac{dz}{dx} = 0$, ſi eae fuerint cognitae, non ſolum numerus radicum realium aequationis $z = 0$ colligitur, ſed etiam limites innoteſcunt, intra quos ſingulae iſtae radices contineantur.

EXEMPLUM.

Sit propoſita iſta aequatio: $x^4 - 14xx + 24x - 12 = 0$ *quae an habeat radices reales & quot quaeritur.*

Aequatio differentialis erit $4x^3 - 28x + 24 = 0$, ſeu $x^3 - 7x + 6 = 0$, cuius radices ſunt 1, 2, & -3, quae ſecundum ordinem magnitudinis diſpoſitae dabunt

$\alpha =$

unde erit

$\alpha = 2$	$\mathfrak{A} = -4$
$\beta = 1$	$\mathfrak{B} = -1$
$\gamma = -3$	$\mathfrak{C} = -129$

Ob $\mathfrak{A}$ negativum ergo aequatio propoſita habebit radicem realem > 2, at ob $\mathfrak{B}$ negativum, neque inter limites 2 & 1, neque inter limites 1 & -3 radicem habebit realem. Cum autem poſito $x = -3$ fiat $z = \mathfrak{C} = -129$, ac ſi ſtatuatur $x = -\infty$, fiat $z = +\infty$ neceſſe eſt, ut radix detur realis inter limites -3 & $-\infty$ contenta. Habebit ergo aequatio propoſita duas radices reales, alteram $x > 2$, alteram $x < -3$; ex quo duae radices erunt imaginariae. Simili modo ergo ex ultimo aequationis propoſitae maximo vel minimo iudicari debet, quo ex primo ſolo. Scilicet ſi aequatio propoſita fuerit ordinis paris, ultimum ſive maximum ſive minimum (erit autem hoc caſu minimum), ſi fuerit negativum radicem realem, ſin affirmativum radicem imaginariam indicat. At pro aequationibus imparium graduum, quia poſito $x = -\infty$ fit $z = -\infty$, ſi ultimum maximum fuerit affirmativum, radix realis, ſin negativum, imaginaria indicatur.

302. Regula ergo pro cognoſcendis radicibus realibus & imaginariis hoc modo commode exprimi poterit. Propoſita aequatione quacunque $z = 0$, conſideretur eius differentialis $\frac{dz}{dx} = 0$, cuius radices reales ſecundum ordinem quantitatis diſpoſitae ſint α, β, γ, δ, &c. tum poſito

	$x =$	$\alpha, \beta, \gamma, \delta, \varepsilon, \zeta,$	&c.
fiat	$z =$	$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F},$	&c.
Iam ſi ſigna ſint:		$- \; + \; - \; + \; - \; +$	&c.

tot aequatio $z = 0$ habebit radices reales, quot habentur litterae α, β, γ, &c. & inſuper unam. Sin autem una ex his litteris maiuſculis non habeat ſignum infraſcriptum, tum binae radices imaginariae indicabuntur. Ita ſi $\mathfrak{A}$ haberet ſignum

 +,

$+$, tum nulla daretur radix intra limites ∞ & β contenta. Si $\mathfrak{B}$ habeat ſignum $-$, nulla dabitur radix inter limites α & γ; & ſi $\mathfrak{C}$ habeat ſignum $+$, nulla erit radix inter limites β & δ, & ita porro. Generatim autem praeter radices imaginarias hoc modo indicatas, aequatio $z = 0$ inſuper tot habebit imaginarias, quot aequatio $\frac{dz}{dx} = 0$.

303. Si eveniat, ut valorum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. aliquis evaneſcat, tum eo loco aequatio $z = 0$ duas habebit radices aequales. Scilicet ſi fuerit $\mathfrak{A} = 0$, tum habebit duas radices ipſi α aequales; ſin ſit $\mathfrak{B} = 0$, duae erunt radices $= \beta$. Hoc enim caſu aequatio $z = 0$ unam habebit radicem communem cum aequatione differentiali $\frac{dz}{dx} = 0$; ſupra autem demonſtravimus, hoc eſſe indicium duarum radicum aequalium. Sin autem aequatio $\frac{dz}{dx} = 0$ duas pluresve radices habeat aequales, tum ſi earum numerus fuerit par, neque maximum neque minimum indicabitur: unde pro praeſenti inſtituto radices aequales numero pares negligi poterunt. Sin autem numerus radicum aequalium aequationis $\frac{dz}{dx} = 0$ fuerit impar, tum omnes praeter unam in formatione iudicii reiiciendae ſunt; niſi forte hoc caſu ipſa quoque functio z evaneſcat. Si enim hoc eveniat aequatio $z = 0$ quoque habebit radices aequales & quidem una plures, quam aequatio $\frac{dz}{dx} = 0$. Sic ſi fuerit $\frac{dz}{dx} = (x - \zeta)^n R$, ita ut haec aequatio habeat n radices aequales ipſi ζ, ſi poſito $x = \zeta$ quoque evaneſcat z, tum aequatio $z = 0$ habebit $n + 1$ radices aequales ipſi ζ.

304. Applicemus haec praecepta ad aequationes ſimpliciores, ac primo quidem a quadratica incipiamus. Sit igitur propo-

propoſita haec aequatio: $z = x^2 - Ax + B = 0$: erit eius differentialis $\frac{dz}{dx} = 2x - A$, qua facta $= 0$, erit $x = \frac{1}{2}A$, ſeu $x = \frac{1}{2}A$. Subſtituatur hic valor loco x, fietque $z = -\frac{1}{4}AA + B = \mathfrak{A}$; unde colligimus, ſi iſte valor ipſius $\mathfrak{A}$ fuerit negativus, hoc eſt ſi ſit $AA > 4B$, aequationem $xx - Ax + B = 0$ habituram eſſe duas radices reales, alteram maiorem quam $\frac{1}{2}A$ alteram minorem. Sin autem valor ipſius $\mathfrak{A}$ fuerit affirmativus ſeu $AA < 4B$, tum ambae aequationis propoſitae radices erunt imaginariae. At ſi fuerit $\mathfrak{A} = 0$ ſeu $AA = 4B$, tum aequatio propoſita habebit duas radices aequales, utramque ſcilicet $= \frac{1}{2}A$. Quae cum ex natura aequationum quadraticarum ſint notiſſima, veritas horum principiorum non mediocriter illuſtratur, ſimulque eorum utilitas in hoc negotio perſpicitur.

305. Progrediamur ergo ad aequationes cubicas ſimili modo inquirendas. Sit ergo propoſita aequatio
$x^3 - Ax^2 + Bx - C = z = 0$: cuius differentialis cum ſit $3xx - 2Ax + B = \frac{dz}{dx}$, ſi haec ponatur $= 0$, fiet

$xx = \frac{2Ax - B}{3}$, cuius aequationis vel ambae radices ſunt imaginariae, vel aequales, vel reales inaequales. Cum igitur hinc ſit $x = \frac{A \pm \sqrt{(A^2 - 3B)}}{3}$, ambae radices erunt imaginariae, ſi fuerit $AA < 3B$: hoc ergo caſu aequatio cubica propoſita unicam habebit radicem realem, cuius alii limites non patent praeter $+\infty$ & $-\infty$. Sint iam ambae radices inter ſe aequales, ſeu $AA = 3B$, erit $x = \frac{A}{3}$. Niſi ergo ſimul fiat $z = 0$, hae duae radices pro nullis reputari debebunt, habebitque aequatio ut ante unicam radicem realem;

ſin autem caſu $x = \frac{A}{3}$ ſimul fiat $z = 0$, quod evenit, ſi fuerit $-\frac{2}{27}A^3 + \frac{1}{3}AB - C = 0$, ſeu $C = \frac{1}{3}AB - \frac{2}{27}A^3$; hoc eſt ſi fuerit $B = \frac{1}{3}A^2$ & $C = \frac{1}{27}A^3$, aequatio habebit tres radices aequales, ſingulas ſcilicet $= \frac{1}{3}A$. Evolvamus nunc tertium caſum, quo ambae radices aequationis differentialis ſunt reales & inter ſe inaequales, quod evenit ſi $AA > 3B$. Sit ergo $AA = 3B + ff$, ſeu $B = \frac{1}{3}AA - \frac{1}{3}ff$, erunt ambae illae radices $x = \frac{A \pm f}{3}$. Fiet ergo $\alpha = \frac{1}{3}A + \frac{1}{3}f$ & $\beta = \frac{1}{3}A - \frac{1}{3}f$. Quaerantur ergo valores ipſius z his reſpondentes $\mathfrak{A}$ & $\mathfrak{B}$; & cum ambae radices contineantur in hac aequatione $xx = \frac{2}{3}Ax - \frac{1}{3}B$, fiet $z = -\frac{1}{3}Axx + \frac{2}{3}Bx - C = -\frac{2}{9}AAx + \frac{1}{9}AB + \frac{2}{3}Bx - C$. Hinc itaque oritur:

$$\mathfrak{A} = -\frac{2}{27}A^3 + \frac{1}{3}AB - \frac{2}{27}A^2 f + \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - C$$
$$\mathfrak{B} = -\frac{2}{27}A^3 + \frac{1}{3}AB + \frac{2}{27}A^2 f - \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 - C$$

ob $B = \frac{1}{3}AA - \frac{1}{3}ff$. Si igitur fuerit $\mathfrak{A}$ quantitas negativa, quod evenit, ſi fuerit $C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$, aequatio $z = 0$ unam habebit radicem realem $> \alpha$, hoc eſt maiorem quam $\frac{1}{3}A + \frac{1}{3}f$. Ponamus ergo eſſe $C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$ ſeu eſſe $C = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 + gg$; atque ut vidimus, aequatio propoſita cubica habebit radicem realem $> \frac{1}{3}A + \frac{1}{3}f$. Quales autem futurae ſint reliquae radices, ex valore $\mathfrak{B}$ intelligetur: erit autem $\mathfrak{B} = \frac{4}{27}f^3 - gg$; qui ſi fuerit affirmativus, aequatio inſuper duas habebit radices reales, priorem intra limites α & β, hoc eſt intra $\frac{1}{3}A + \frac{1}{3}f$ & $\frac{1}{3}A - \frac{1}{3}f$ contentam, alteram vero minorem quam $\frac{1}{3}A - \frac{1}{3}f$. Sin autem fuerit $gg > \frac{4}{27}f^3$, ſeu $\mathfrak{B}$ negativum, tum aequatio habebit duas radices imaginarias. At ſi fuerit $\mathfrak{B} = 0$ ſeu $\frac{4}{27}f^3 = gg$ tum duae radices evadent aequales, utraque $= \beta = \frac{1}{3}A - \frac{1}{3}f$. Denique ſi ſit valor ipſius $\mathfrak{A}$ affirmativus ſeu $C < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$, tum aequatio duas habebit radices imaginarias, tertiaque erit realis & $< \frac{1}{3}A - \frac{1}{3}f$. Atque ſi ſit valor ipſius $\mathfrak{A} = 0$, duae erunt radices aequales $= \alpha$, manente tertia $< \frac{1}{3}A - \frac{1}{3}f$.

306.

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306. Quo igitur aequationis cubicae $x^3 - Ax^2 + Bx - C = 0$ omnes tres radices ſint reales, requiruntur tres conditiones. Primo ut ſit $B < \frac{1}{3}AA$: ſit ergo $B = \frac{1}{3}AA - \frac{1}{3}ff$. Secundo ut ſit $C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$. Tertio ut ſit $C < \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$. Quae duae poſteriores conditiones eo redeunt, ut C contineatur intra hos limites $\frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$ & $\frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$ ſeu intra hos limites $\frac{1}{27}(A+f)^2(A-2f)$ & $\frac{1}{27}(A-f)^2(A+2f)$. Quod ſi ergo harum conditionum unica deſit aequatio duas habebit radices imaginarias. Sic ſi fuerit $A = 3$, $B = 2$, erit $\frac{1}{3}ff = \frac{1}{3}AA - B = 1$ & $ff = 3$; unde iſta aequatio: $x^3 - 3xx + 2x - C = 0$ omnes radices reales habere nequit, niſi C contineatur intra limites $-\frac{2\sqrt{3}}{9}$ & $+\frac{2\sqrt{3}}{9}$. Quare ſi fuerit vel $C < -\frac{2\sqrt{3}}{9}$ ſeu $C < -0,3849$, vel $C > +\frac{2\sqrt{3}}{9}$ ſeu $C > 0,3849$ aut coniunctim $CC > \frac{4}{27}$, aequatio unicam habebit radicem realem.

307. Quoniam in omni aequatione ſecundus terminus tolli poteſt, ponamus eſſe $A = 0$, ita ut habeamus hanc aequationem cubicam $x^3 + Bx - C = 0$. Ut igitur huius aequationis omnes tres radices ſint reales, neceſſe eſt ut primo ſit $B < 0$, ſeu B debet eſſe quantitas negativa. Sit ergo $B = -kk$, erit $ff = 3kk$, atque inſuper requiritur, ut quantitas C contineatur intra hos limites $-\frac{2}{27}f^3$ & $+\frac{2}{27}f^3$; hoc eſt inter hos $-\frac{2}{9}kk\sqrt{3kk}$ & $+\frac{2}{9}kk\sqrt{3kk}$. Erit ergo $CC < \frac{4}{27}k^6$, ſeu $CC < -\frac{4}{27}B^3$. Unica ergo conditione natura aequationum cubicarum, quae omnes tres radices habeant reales comprehendi poterit, dum dicemus eſſe oportere $4B^3 + 27CC$ quantitatem negativam. Sic enim iam poſtulatur, ut ſit B quantitas negativa, quia alioquin $4B^3 + 27CC$ negativum fieri non poſſet. Quocirca generatim affirmamus, aequationem $x^3 + Bx \pm C = 0$ omnes tres radices habituram eſſe reales, ſi fuerit $4B^3 + 27CC$ quantitas negativa. Sin autem haec quantitas fuerit affirmativa, tum unicam fore realem, reliquas binas imaginarias; at ſi

fiat

fiat $4B^3+27CC=0$, tum omnes quidem radices futuras eſſe reales, at binas inter ſe aequales.

308. Progrediamur ad aequationes biquadratas, in quibus etiam ſecundum terminum deeſſe ponamus. Sit ergo $x^4+Bx^2-Cx+D=0$. Statuamus $x=\frac{1}{u}$, eritque $1+Bu^2-Cu^3+Du^4=0$, cuius aequatio differentialis eſt $2Bu-3Cu^2+4Du^3=0$, quae unam habet radicem $u=0$, tum vero erit $uu=\frac{6Cu-4B}{8D}$ & $u=\frac{3C\pm\surd(9CC-32BD)}{8D}$. Ut igitur omnes quatuor radices ſint reales, primo requiritur, ut ſit $9CC>32BD$. Ponamus ergo eſſe $9CC=32BD+9ff$, erit $u=\frac{3C\pm 3f}{8D}$. Hic C ſemper pro quantitate affirmativa ſumere poterimus, niſi enim talis fuerit, ponendo $u=-v$ talis evadet. Mox autem demonſtrabimus omnes radices reales eſſe non poſſe, niſi ſit B quantitas negativa. Sit ergo $B=-gg$, eritque $9CC=9ff-32ggD$, & $u=\frac{3C\pm 3f}{8D}$. Atque duo caſus erunt perpendendi, prout D ſit quantitas affirmativa vel negativa.

I. Sit D quantitas affirmativa, eritque $f>C$, ac tres ipſius u radices ſecundum quantitatis ordinem diſpoſitae erunt 1°; $u=\frac{3C+3f}{8D}$, 2°; $u=0$, 3°; $u=\frac{3C-3f}{8D}$. Aequatio autem $u^4-\frac{Cu^3}{D}+\frac{Bu^2}{D}+\frac{1}{D}=0$, his valoribus loco u ſubſtitutis dabit ſequentes tres valores:

$$\mathfrak{A}=\frac{27(C+f)^3\,(C-3f)}{4096\,D^4}+\frac{1}{D}\;;\qquad \mathfrak{B}=\frac{1}{D}\;;$$

$$\mathfrak{C}=\frac{27(C-f)^3\,(C+3f)}{4096\,D^4}+\frac{1}{D}$$

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quorum primus ac tertius debet eſſe negativus: uterque quidem ob C affirmativum & $C < f$ fit minor quam $\frac{1}{D}$: oportet itaque eſſe $\frac{1}{D} < \frac{27(C+f)^3(3f-C)}{4096 D^4}$ & $\frac{1}{D} < \frac{27(f-C)^3(C+3f)}{4096 D^4}$, ſeu $4096 D^3 < 27(f+C)^3(3f-C)$ & $4096 D^3 < 27(f-C)^3(C+3f)$. At prior quantitas ſemper longe maior eſt poſteriori; unde ſufficit, ſi fuerit $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, exiſtente $B = \frac{9CC-9ff}{32 D}$ & $f > C$, atque $D > 0$. Si igitur fuerit D quantitas affirmativa, C affirmativa, B negativa, ut ſit $f > C$, atque $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, hoc eſt $D < \frac{3}{16}(f-C)\sqrt[3]{(3f+C)}$, tum aequatio omnes radices habebit reales. Sin autem fuerit $D > \frac{3}{16}(f-C)\sqrt[3]{(3f+C)}$, attamen $D < \frac{3}{16}(f+C)\sqrt[3]{(3f-C)}$; tum duae radices erunt reales & duae imaginariae. At ſi adeo fuerit $D > \frac{3}{16}(f+C)\sqrt[3]{(3f-C)}$; tum omnes quatuor radices erunt imaginariae.

II. Sit D quantitas negativa puta $= -F$, manente C affirmativa ac B negativa, ob $B = \frac{9CC-9ff}{32 D} = \frac{9ff-9CC}{32 F}$, erit $C > f$. Cum igitur ſit $u = \frac{3C \pm 3f}{8D} = \frac{-3C \mp 3f}{8F}$ tres valores ipſius u ſecundum ordinem magnitudinis diſpoſiti erunt 1°, $u = 0$; 2°, $u = \frac{-3C+3f}{8F}$; 3°, $u = \frac{-3C-3f}{8F}$, qui dabunt ſe-

quen-

quentes valores $\mathfrak{A} = -\frac{1}{F}$

$$\mathfrak{B} = \frac{27(C-f)^3\,(C+3f)}{4096\,F^4} - \frac{1}{F}$$

$$\mathfrak{C} = \frac{27(C+f)^3\,(C-3f)}{4096\,F^4} - \frac{1}{F}$$

Cum igitur $\mathfrak{A}$ ſit quantitas negativa, aequatio iam certo unam, ac propterea quoque duas habebit radices reales. Ut autem omnes radices ſint reales, oportet ut $\mathfrak{B}$ ſit quantitas affirmativa, ideoque $27(C-f)^3(C+3f) > 4096\,F^3$: tum vero neceſſe eſt, ut ſit $\mathfrak{C}$ quantitas negativa ſeu $27(C+f)^3\,(C-3f) < 4096\,F^3$. Quocirca ut omnes radices fiant reales, requiritur ut F^3 contineatur intra hos limites $\frac{27}{4096}(C+f)^3(C-3f)$ & $\frac{27}{4096}(C-f)^3(C+3f)$ ſeu ut F contineatur intra limites $\frac{3}{16}(C+f)\sqrt[3]{(C-3f)}$ & $\frac{3}{16}(C-f)\sqrt[3]{(C+3f)}$; & niſi F contineatur intra hos limites, duae radices erunt imaginariae.

III. Ponamus iam B eſſe quantitatem affirmativam, & D pariter affirmativam, ob $B = \frac{9CC-9ff}{32D}$, erit $C > f$, & cum ſit $u = \frac{3C \pm 3f}{8D}$, radices ordine magnitudinis diſpoſitae erunt 1°, $u = \frac{3(C+f)}{8D}$; 2°, $u = \frac{3(C-f)}{8D}$ & 3°, $u = 0$. unde ſequentes oriuntur valores:

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096\,D^4} + \frac{1}{D};$$

$$\mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096\,D^4} + \frac{1}{D}; \quad \mathfrak{C} = \frac{1}{D};$$

ubi

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ubi cum $\mathfrak{C}$ ſit quantitas affirmativa, certo duae radices erunt imaginariae. Sin autem fuerit $\mathfrak{A}$ negativum quod evenit, ſi $4096\,D^3 < 27\,(C+f)^3\,(3f-C)$, duae radices erunt reales: ſin fuerit $4096\,D^3 > 27\,(C+f)^3\,(3f-C)$, tum omnes quatuor radices erunt imaginariae.

IV. Maneat B affirmativum, ſit autem D negativum $= -F$, ob $B = \frac{9ff - 9CC}{32\,F}$, erit $f > C$ & ob $u = \frac{-3C \mp 3f}{8F}$, tres ipſius u radices ſecundum ordinem magnitudinis diſpoſitae erunt 1°, $u = \frac{3(f-C)}{8\,F}$; 2°, $u = 0$; & 3°, $u = \frac{-3(C+f)}{8F}$, unde iſti valores naſcuntur.

$$\mathfrak{A} = \frac{-27(f-C)^3\,(C+3f)}{4096\,F^4} - \frac{1}{F}$$

$$\mathfrak{B} = - \frac{1}{F}$$

$$\mathfrak{C} = \frac{-27(C+f)^3\,(3f-C)}{4096\,F^4} - \frac{1}{F}$$

ubi ob $\mathfrak{A}$ & $\mathfrak{C}$ negativa aequatio certo duas habet radices reales, at ob $\mathfrak{B}$ negativum, duae radices erunt imaginariae.

309. Si igitur ponamus litteras B, C, D quantitates affirmativas denotare, ſequentes oriuntur caſus diverſi diiudicandi, qui ob $f = \sqrt{(CC - \frac{32}{9}BD)}$ huc redeunt.

I. Si aequatio ſit $x^4 - Bx^2 \pm Cx + D = 0$. Omnes radices erunt reales, ſi fuerit

$$D < \tfrac{1}{16}[\sqrt{(CC + \tfrac{32}{9}BD)} - C]\sqrt[3]{[3\sqrt{(CC + \tfrac{32}{9}BD)} + C]}.$$

Duae radices erunt reales, duaeque imaginariae, ſi fuerit

$$D > \tfrac{1}{16}[\sqrt{(CC + \tfrac{32}{9}BD)} - C]\sqrt[3]{[3\sqrt{(CC + \tfrac{32}{9}BD)} + C]}$$

at $D < \tfrac{1}{16}[\sqrt{(CC + \tfrac{32}{9}BD)} + C]\sqrt[3]{[3\sqrt{(CC + \tfrac{32}{9}CD)} - C]}$

Omnes autem radices erunt imaginariae, ſi fuerit

$$D > \tfrac{3}{16}[\sqrt{(CC + \tfrac{32}{9}BD)} + C]\sqrt[3]{[3\sqrt{(CC + \tfrac{32}{9}BD)} - C]}.$$

II. Si aequatio ſit $x^4 - Bx^2 \pm Cx - D = 0$. Duae radices ſemper ſunt reales, reliquae binae quoque erunt reales, ſi quantitas D contineatur intra hos limites.

$$D > \tfrac{3}{16}[\sqrt{(CC - \tfrac{32}{9}BD)} + C]\sqrt[3]{[C - 3\sqrt{(CC - \tfrac{32}{9}BD)}]}$$

$$D < \tfrac{3}{16}[C - \sqrt{(CC - \tfrac{32}{9}BD)}]\sqrt[3]{[C + 3\sqrt{(CC - \tfrac{32}{9}BD)}]}$$

niſi autem D contineatur intra hos limites, duae reliquae radices erunt imaginariae.

III. Si aequatio ſit $x^4 + Bx^2 \pm Cx + D = 0$. Duae radices ſemper erunt imaginariae. Reliquae vero duae erunt reales, ſi fuerit

$$D < \tfrac{3}{16}[\sqrt{(CC - \tfrac{32}{9}BD)} + C]\sqrt[3]{[3\sqrt{(CC - \tfrac{32}{9}BD)} - C]}$$

Reliquae vero duae quoque erunt imaginariae, ſi fuerit

$$D > \tfrac{3}{16}[\sqrt{(CC - \tfrac{32}{9}BD)} + C]\sqrt[3]{[3\sqrt{(CC - \tfrac{32}{9}BD)} - C]}$$

IV. Si aequatio ſit $x^4 + Bx^2 \pm Cx - D = 0$. Huius aequationis duae radices ſemper erunt reales, duae reliquae vero ſemper imaginariae.

EXEMPLUM I.

Si proponatur haec aequatio $x^4 - 2xx + 3x + 4 = 0$ *quaeratur natura radicum, utrum ſint reales an imaginariae.*

Quia hoc exemplum ad caſum primum pertinet, eſt $B = +2$; $C = 3$ & $D = 4$; unde $CC + \frac{32}{9}BD = 9 + \frac{32 \cdot 8}{9} = \frac{337}{9}$ & $\sqrt{(CC + \frac{32}{9}BD)} = \frac{\sqrt{337}}{3}$ unde conditiones ut omnes radices ſint reales, ſunt

$$4 < \frac{3}{16}\left(3 + \frac{\sqrt{337}}{3}\right)\sqrt[3]{(\sqrt{337} - 3)} = \frac{1}{16}(9 + \sqrt{337})\sqrt[3]{(\sqrt{337} - 3)}$$

$$4 < \frac{3}{16}\left(\frac{\sqrt{337}}{3} - 3\right)\sqrt[3]{(\sqrt{337} + 3)} = \frac{1}{16}(\sqrt{337} - 9)\sqrt[3]{(\sqrt{337} + 3)}$$

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Adhibitis, approximationibus examinari debet ergo, utrum ſit $4 < \frac{69}{16}$ & $4 < \frac{24}{16}$; quare cum prior tantum conditio locum habeat, aequatio habebit duas radices reales, & duas imaginarias.

EXEMPLUM II.

Propoſita ſit haec aequatio: $x^4 - 9xx + 12x - 4 = 0$.

Quae cum pertineat ad caſum ſecundum, duas habebit radices reales. Ad reliquarum naturam inveſtigandam, ob $B = 9$, $C = 12$ & $D = 4$, erit $\surd(CC - \frac{32}{9}BD) = \surd(144 - 32.4) = 4$. Ideoque videndum eſt, utrum ſit

$$4 > \frac{3}{16}.16\sqrt[3]{0} \quad \text{hoc eſt } 4 > 0$$

$$\& \; 4 < \frac{3}{16}.8\sqrt[3]{24^2} \quad \text{hoc eſt } 4 < 3\sqrt[3]{3}$$

quorum utrumque cum eveniat, aequatio propoſita quatuor habebit radices reales.

EXEMPLUM III.

Propoſita ſit haec aequatio: $x^4 + xx - 2x + 6 = 0$.

Quae cum pertineat ad caſum tertium, duae radices certo erunt imaginariae. Tum vero eſt $B = 1$; $C = 2$ & $D = 6$, ideoque $\surd(CC - \frac{32}{9}BD) = \surd\left(4 - \frac{64}{3}\right)$, quae quantitas cum ſit imaginaria, & duae reliquae radices certo erunt imaginariae.

EXEMPLUM. IV.

Sit propoſita aequatio haec: $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$.

Eliminetur primo ſecundus terminus, ſubſtituendo $x = y + 1$ fiet.

$$
\begin{array}{rcl}
x^4 & = & y^4 + 4y^3 + 6yy + 4y + 1 \\
-4\ x^3 & = & -4y^3 - 12y^2 - 12y - 4 \\
+8\ x^2 & = & +8y^2 + 16y + 8 \\
-16\,x & = & -16y - 16 \\
+20 & = & +20 \\
\hline
\text{Ergo} & & y^4 + 2yy - 8y + 9 = 0
\end{array}
$$

quae cum pertineat ad caſum tertium, duas radices habebit imaginarias. Tum vero ob $B = 2$, $C = 8$, $D = 9$, erit $\sqrt{(CC - \frac{32}{9}.\ BD)} = \sqrt{(64 - 64)} = 0$. Comparetur ergo $D = 9$ cum $\frac{3}{16}.\ 8\sqrt[3]{-8} = -3$. Cum ergo ſit $D = 9 > -3$, etiam duae reliquae radices erunt imaginariae.

EXEMPLUM V.

Sit propoſita haec aequatio: $x^4 - 4x^3 - 7x^2 + 34x - 24 = 0$ *cuius radices conſtant eſſe* $1, 2, 4$ & -3.

Quod ſi autem regulas applicemus, ſublato ſecundo termino ponendo $x = y + 1$ fiet: $y^4 - 13yy + 12y + 0 = 0$ quae cum caſu ſecundo comparata dat $B = 13$, $C = 12$; $D = 0$. Debet ergo eſſe $D > \frac{3}{16}.\ 24.\ \sqrt[3]{-24}$; ſeu $0 > -9\sqrt[3]{3}$ & $D < 0$; cum igitur D non ſit maius quam 0, aequatio quatuor radices reales habere indicatur. Si enim ſit $D = 0$, altera aequatio abit in $D < \frac{3}{16}\left(\frac{16\,BD}{9C}\right)\sqrt[3]{4C}$ ideoque $1 < \frac{B}{3C}\sqrt[3]{4C}$, ſeu $27CC < 4B^3$: eſt vero $27.\ 144 < 4.\ 13^3$ ſeu $36.\ 27 < 13^3$.

310. Opus foret maxime difficile, ſi ſimile iudicium ad aequationes altiorum graduum transferre vellemus, propterea quod aequationum differentialium radices plerumque exhiberi non poſſunt; quoties autem has radices aſſignare licet, ex traditis principiis facile colligitur, quot aequatio propoſita habeat radices reales & imaginarias. Hinc omnis aequationis, quae tantum ex tribus terminis conſtat, radices, u-

trum

trum ſint reales an imaginariae, definiri poterunt. Sit enim propoſita haec aequatio generalis:

$$x^{m+n} + Ax^n + B = 0 = z$$

Sumatur eius differentialis $\frac{dz}{dx} = (m+n)x^{m+n-1} + nAx^{n-1}$, qua nihilo aequali poſita, erit primo $x^{n-1} = 0$; unde ſi n fuerit impar numerus, nulla radix maximum minimumve exhibens oritur: ſin autem ſit n numerus par, una radix in computum ducenda erit $x = 0$. Tum vero erit $(m+n)x^m + nA = 0$; quae aequatio, ſi m ſit numerus par, & A affirmativa quantitas, nullam habet radicem realem. Hinc ſequentes caſus erunt expendendi.

I. Sit m numerus par & n numerus impar, & radix $x = 0$ non valebit. Si igitur fuerit A quantitas affirmativa, nulla prorſus habebitur radix maximum minimumve exhibens; unde ob $m+n$ numerum imparem aequatio propoſita unicam habebit radicem realem. Sin autem fuerit A quantitas negativa; puta $A = -E$, erit $x = \pm\sqrt[m]{\frac{nE}{m+n}}$: unde $\alpha = +\sqrt[m]{\frac{nE}{m+n}}$ & $\beta = -\sqrt[m]{\frac{nE}{m+n}}$. Ex quibus valoribus fit:

$\mathfrak{A} = (x^m - E)x^n + B = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$ atque $\mathfrak{B} = +\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$. Si igitur fuerit $\mathfrak{A}$ quantitas negativa, ſeu $\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} > B$, aequatio unam habebit radicem realem $> \alpha$.

Si inſuper fuerit $B > -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m}$, hoc eſt ambas conditiones in unam complectendo, ſi fuerit $(m+n)^{m+n}B^m < m^m n^n E^{m+n}$, tum aequatio tres habebit

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bit radices reales: &, niſi haec conditio locum habeat, aequationis unica radix erit realis. Valent haec de aequatione $x^{m+n} - Ex^n + B = o$, ſi fuerit m numerus impar: ubi ſi E fuerit numerus negativus, aequatio ſemper unicam radicem habebit realem.

II. Sint ambo numeri m & n impares, ut ſit $m+n$ numerus par, nullaque radix $x = o$ in computum veniat. Quia eſt $(m+n)x^m + nA = o$, erit $x = -\sqrt[m]{\frac{nA}{m+n}}$, quae unica radix ſi ſit $= \alpha$, fiet $\mathfrak{A} = \frac{mA}{m+n}\alpha^n + B = -\frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B$. Qui valor ſi fuerit negativus, aequatio propoſita duas habebit radices reales, contra nullam. Aequatio ergo propoſita $x^{m+n} + Ax^n + B = o$ duas habebit radices reales, ſi fuerit $m^m n^n A^{m+n} > (m+n)^{m+n} B^m$; ſin fuerit $m^m n^n A^{m+n} < (m+n)^{m+n} B^m$, nulla prorſus radix erit realis.

III. Sint ambo numeri m & n pares, erit $m+n$ pariter numerus par: unaque radix $x = o$ maximum minimumve praebebit: quae erit unica, ſi A fuerit quantitas affirmativa, unde facto $\alpha = o$, erit $\mathfrak{A} = B$. Quare ſi fuerit B quoque quantitas affirmativa, aequatio nullam habebit radicem realem; ſin autem B ſit quantitas negativa, duae habebuntur radices reales, neque plures, ſi quidem A fuerit quantitas affirmativa. At ponamus eſſe A quantitatem negativam ſeu $A = -E$, erit $x = \pm\sqrt[m]{\frac{nE}{m+n}}$; habebimuſque tria maxima vel minima: nempe $\alpha = +\sqrt[m]{\frac{nE}{m+n}}$; $\beta = o$; $\gamma = -\sqrt[m]{\frac{nE}{m+n}}$. Quibus ipſius $z = x^{m+n} - Ex^n + B = o$ reſpondent valores

$$\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B;$$

$$\mathfrak{B} = B;$$

$$\mathfrak{C} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B.$$

Si

Si igitur B ſit quantitas negativa, ob $\mathfrak{A}$ & $\mathfrak{C}$ negativas, aequatio duas tantum habebit radices reales, propterea quod quoque $\mathfrak{B} = B$ ſit negativum. At ſi B fuerit quantitas affirmativa, aequatio quatuor habebit radices reales, ſi ſit $(m+n)^{m+n} B^m < m^m n^n E^{m+n}$. Nullam autem habebit radicem realem, ſi fuerit $(m+n)^{m+n} B^m < m^n n^n E^{m+n}$.

IV. Sit m numerus impar & n numerus par: atque radix $x = 0$ dabit maximum vel minimum. Praeterea vero erit $x = -\sqrt[m]{\frac{nA}{m+n}}$. Si ergo A ſit numerus affirmativus, fiet $\alpha = 0$ & $\beta = -\sqrt[m]{\frac{nA}{m+n}}$; hincque $\mathfrak{A} = B$, & $\mathfrak{B} = \frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B$. Quare ſi ſit B quantitas negativa, puta $B = -F$, atque inſuper fuerit $m^m n^n A^{m+n} > (m+n)^{m+n} F^m$, aequatio tres habebit radices reales; contra unica tantum erit realis. Sin autem ſit A quantitas negativa puta $A = -E$, fiet $x = +\sqrt[m]{\frac{nE}{m+n}}$ & $\alpha = \sqrt[m]{\frac{nE}{m+n}}$ & $\beta = 0$, quibus reſpondent $\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$ & $\mathfrak{B} = B$. Quare aequatio tres habebit radices reales; ſi fuerit B quantitas affirmativa, & $m^m n^n E^{m+n} > (m+n)^{m+n} B^m$, quae proprietas niſi locum inveniat, aequatio unicam habebit radicem realem.

311. Sint omnes coefficientes $= 1$, atque denotantibus μ & ν numeros integros, aequationes ſequentes ita diiudicabuntur:

$x^{2\mu+2\nu-1} + x^{2\nu-1} \pm 1 = 0$ unicam habebit radicem realem:

$x^{2\mu+2\nu-1} - x^{2\nu-1} \pm 1 = 0$, tres habebit radices reales, ſi fuerit $(2\mu+2\nu-1)^{2\mu+2\nu-1} < (2\mu)^{2\mu}(2\nu-1)^{2\nu-1}$, quod cum nunquam fieri poſſit, aequatio ſemper unicam radicem realem habebit:

$x^{2\mu}$

$x^{2\mu+2\nu} \pm x^{2\nu-1} - 1 = 0$	duas habet radices reales.
$x^{2\mu+2\nu} \pm x^{2\nu-1} + 1 = 0$	nullam habet radicem realem.
$x^{2\mu+2\nu} \pm x^{2\nu} + 1 = 0$	nullam habet radicem realem.
$x^{2\mu+2\nu} \pm x^{2\nu} - 1 = 0$	duas habet radices reales.
$x^{2\mu+2\nu+1} + x^{2\nu} \pm 1 = 0$	unicam habet radicem realem.
$x^{2\mu+2\nu+1} - x^{2\nu} \pm 1 = 0$	unicam habet radicem realem.

Ceterum quia in caſu tertio ambo exponentes ſunt pares: is ponendo $xx = y$ ad formam ſimpliciorem reduci poteſt, ideoque hic caſus praetermitti poſſet. Quo facto affirmari poterit, nullam aequationem tribus terminis conſtantem plures tribus habere poſſe radices reales.

EXEMPLUM.

Quaerantur caſus, quibus aequatio haec $x^5 \pm Ax^2 \pm B = 0$ *tres habeat radices reales.*

Quia haec aequatio pertinet ad caſum quartum, patet quantitates A & B, eſſe debere ſignis contrariis affectas. Quare niſi huiuſmodi habeat formam, unicam habebit radicem realem: ſin autem aequatio propoſita fuerit huiuſmodi $x^5 \pm Ax^2 \mp B = 0$, quo ea habeat tres radices reales, neceſſe eſt ut ſit $3^3 2^2 A^5 > 5^5 B^3$ ſeu $A^5 > \frac{3125}{108} B^3$. Quod ſi ergo fuerit $B = 1$, oportet eſſe, $A^5 > \frac{3125}{108}$, ſeu $A > 1,960132$. Si ergo ſit $A = 2$ iſta aequatio $x^5 - 2x^2 + 1 = 0$ tres habet radices reales, quarum cum una ſit $x = 1$; ſequitur hanc aequationem biquadratam $x^4 + x^3 + x^2 - x - 1 = 0$ duas habere radices reales. Quod quidem tum ex his datis praeceptis intelligi poteſt, tum ex iis, quae in libro ſuperiori ſunt demonſtrata, manifeſtum eſt, ubi oſtendimus, quamvis aequationem paris gradus, cuius terminus abſolutus ſit numerus negativus, habere ſemper duas radices reales.

312. Ex his principiis quoque aequationes, quae conſtant quatuor terminis, diiudicari poterunt, dummodo aequationis

tionis differentialis radices commode exhiberi queant, quod evenit, ſi exponentes ipſius x vel in tribus anterioribus, vel in tribus poſterioribus terminis ſint in arithmetica progreſſione. Cum autem haec diiudicatio in genere ſuſcepta ad plures perducatur caſus, eam in nonnullis exemplis abſolvamus.

EXEMPLUM I.

Sit propoſita haec aequatio $x^7 - 2x^5 + x^3 - a = 0$.

Facto $z = x^7 - 2x^5 + x^3 - a$, erit $\frac{dz}{dx} = 7x^6 - 10x^4 + 3xx$, quo valore nihilo aequali poſito fiet primo $xx = 0$, qui duplex valor pro nullo reputandus. Tum vero erit $7x^4 = 10x^2 - 3$; unde fit $x^2 = \frac{5 \pm 2}{7}$; & quatuor valores pro x emergent, qui ſecundum magnitudinem ordinati ſequentes pro z praebebunt valores:

$\alpha = 1$	$\mathfrak{A} = -a$
$\beta = +\sqrt{\tfrac{3}{7}}$	$\mathfrak{B} = \frac{48}{343}\sqrt{\tfrac{3}{7}} - a$
$\gamma = -\sqrt{\tfrac{3}{7}}$	$\mathfrak{C} = \frac{-48}{343}\sqrt{\tfrac{3}{7}} - a$
$\delta = -1$	$\mathfrak{D} = -a.$

Si ergo fit a numerus affirmativus, erit vel $a > \frac{48}{343}\sqrt{\tfrac{3}{7}}$, vel $a < \frac{48}{343}\sqrt{\tfrac{3}{7}}$, priori caſu ob $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, omnes negativas, aequatio propoſita unicam habebit radicem realem $x > 1$. Poſteriori caſu ſi $a < \frac{48}{343}\sqrt{\tfrac{3}{7}}$ aequatio tres habebit radices reales, primam > 1, ſecundam contentam inter limites 1 & $\sqrt{\tfrac{3}{7}}$, & tertiam intra limites $+\sqrt{\tfrac{3}{7}}$ & $-\sqrt{\tfrac{3}{7}}$.

Sin a ſit quantitas negativa ponendo $x = -y$, aequatio perducetur ad formam priorem. Quo ergo aequatio propoſita

poſita tres habeat radices reales, neceſſe eſt ut ſit $a < 0,0916134$ vel $a < \frac{1}{11}$.

EXEMPLUM II.

Sit propoſita haec aequatio: $ax^8 - 3x^6 + 10x^3 - 12 = 0$. Quia hic exponentes trium poſteriorum terminorum ſunt in arithmetica progreſſione, ponatur $x = \frac{1}{y}$ atque aequatio tranſmutabitur in hanc:

$a - 3y^2 + 10y^5 - 12y^8 = 0$, ponatur ergo

$z = 12y^8 - 10y^5 + 3y^2 - a = 0$, eritque differentiando

$\frac{dz}{dy} = 96y^7 - 50y^4 + 6y = 0$, ex qua aequatione primo fit $y = 0$; tum vero erit $y^6 = \frac{50y^3 - 6}{96}$ & $y^3 = \frac{25 \pm 7}{96}$, ideoque vel $y = \sqrt[3]{\frac{1}{3}}$ vel $y = \sqrt[3]{\frac{3}{16}}$. His ergo tribus radicibus ſecundum magnitudinem diſpoſitis, reſpondentes ipſius z valores ita ſe habebunt:

$\alpha = \sqrt[3]{\frac{1}{3}}$	$\mathfrak{A} = \sqrt[3]{\frac{1}{9}} - a$
$\beta = \sqrt[3]{\frac{3}{16}}$	$\mathfrak{B} = \frac{99}{64}\sqrt[3]{\frac{9}{256}} - a = \frac{99}{256}\sqrt[3]{\frac{9}{4}} - a$
$\gamma = 0$	$\mathfrak{C} = -a.$

Quodſi ergo fuerit $a > \sqrt[3]{\frac{1}{9}}$, aequatio propoſita duas habebit radices reales, alteram $> \sqrt[3]{\frac{1}{3}}$, alteram < 0: at praeter has inſuper habebit duas radices reales, ſi ſimul fuerit $\mathfrak{B}$ quantitas affirmativa, hoc eſt, ſi fuerit $a < \frac{99}{256}\sqrt[3]{\frac{9}{4}}$. Quamobrem

obrem aequatio propoſita quatuor habebit radices reales, ſi quantitas a contineatur intra limites $\sqrt[3]{\frac{1}{9}}$ & $\frac{99}{256}\sqrt[3]{\frac{9}{4}}$; qui limites proxime ſunt: 0,48075 & 0,50674. Poſito ergo $a = \frac{1}{2}$, haec aequatio $x^8 - 6x^6 + 20x^3 - 24 = 0$ quatuor habet radices reales intra limites ∞; $\sqrt[3]{\frac{16}{3}}$; $\sqrt[3]{3}$; 0; $-\infty$; ergo tres erunt affirmativae & una negativa.

CAPUT XIIII.

DE CRITERIIS RADICUM IMAGINARIARUM.

313.

In Capite praecedenti modum exhibuimus naturam radicum cuiuſque aequationis explorandi, ita ut eius beneficio, ſi proponatur aequatio quaecunque, inveniri poſſit, quot ea radices habeat reales, & quot imaginarias. Plerumque quidem haec inveſtigatio difficillime inſtituitur, cum aequatio differentialis ita eſt comparata, ut eius radices exhiberi nequeant. Quanquam autem his caſibus eadem operatio ad aequationem differentialem ipſam accommodari, eiuſque radicum natura ex ipſius differentiali indagari, hincque illius radices proxime aſſignari poſſent; tamen labor nimium ſaepiſſime fieret moleſtus. Quamobrem in hoc negotio ſaepenumero ſufficit eiuſmodi criteria noſſe, ex quorum praeſentia tuto concludi poſſit, ineſſe in aequatione propoſita radices imaginarias; etiamſi ex eorum abſentia viciſſim inferri nequeat, omnes prorſus radices eſſe reales. Quae cognitio etſi eſt imperfecta, tamen frequenter uſu non deſtituitur; quocirca his criteriis explicandis praeſens caput deſtinavimus.

314. In capite igitur praecedenti vidimus, ſi aequatio quaecunque:

$$z = x - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$$

omnes radices habeat reales, tum etiam eius differentialem

$$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \&c. = 0$$

omnes ſuas radices habituram eſſe reales. Simul vero oſtendimus, etiamſi aequatio differentialis omnes habeat radices reales; tamen inde non ſequi, ipſius aequationis propoſitae omnes radi-

radices futuras eſſe reales. Interim tamen, ſi aequatio differentialis habeat radices imaginarias, tum ſemper recte concludimus, aequationem ipſam propoſitam ad minimum totidem habere debere radices imaginarias. Ad minimum dico: fieri enim poteſt, ut ipſa aequatio plures habeat radices imaginarias. Hoc ergo modo ex aequatione differentiali plus concludi non poteſt, quam, ſi ea habeat radices imaginarias, ipſam propoſitam aequationem eiuſmodi radices quoque habere debere, & quidem ad minimum totidem.

315. Si aequatio propoſita multiplicetur per poteſtatem quamcumque x^m, denotante m numerum integrum affirmativum; tum quia haec nova aequatio omnes radices habebit reales, ſi quidem propoſitae radices omnes fuerint reales: tum quoque eius differentialis, poſtquam per x^{m-1} fuerit diviſa, radices erunt reales omnes. Hinc ſi haec aequatio

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$$

omnes radices habeat reales, tum quoque iſta aequatio

$$(m+n)x^n - (m+n-1)Ax^{n-1} + (m+n-2)Bx^{n-2} - \&c. = 0$$

omnes radices habebit reales. Ob eandem rationem, ſi haec multiplicetur per x^k & denuo differentietur aequatio reſultans

$$(m+n)(k+n)x^n - (m+n-1)(k+n-1)Ax^{n-1} + (m+n-2)(k+n-2)Bx^{n-2} - \&c. = 0$$

omnes adhuc radices habebit reales: ſicque quouſque libuerit, ulterius progredi licet. Sin autem huiuſmodi aequatio radices imaginarias habere deprehendatur, tum ſimul certum erit, ipſam aequationem propoſitam ſaltem totidem radices imaginarias eſſe habituram.

316. Si aequatio propoſita antequam differentietur, per nullam poteſtatem ipſius x multiplicetur, tum iudicium ad aequationem uno gradu inferiorem deducitur. Ita ſi aequatio propoſita

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

omnes radices habeat reales, tum quoque eius differentiales omnium ordinum omnes radices habebunt reales. Quare & ſequentium aequationum omnium radices erunt reales:

nx

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$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \&c. = 0$$
$$n(n-1)x^{n-2} - (n-1)(n-2)Ax^{n-3} + (n-2)(n-3)Bx^{n-4} - \&c. = 0$$
$$n(n-1)(n-2)x^{n-3} - (n-1)(n-2)(n-3)Ax^{n-4} + \&c. = 0$$
$$n(n-1)(n-2)(n-3)x^{n-4} - (n-1)(n-2)(n-3)(n-4)Ax^{n-5} + \&c. = 0 \qquad \&c.$$

quae aequationes ad ſequentes formas revocantur:

$$x^{n-1} - \frac{n-1}{n}Ax^{n-2} + \frac{(n-1)(n-2)}{n(n-1)}Bx^{n-3} - \frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)}Cx^{n-4} + \&c. = 0$$
$$x^{n-2} - \frac{n-2}{n}Ax^{n-3} + \frac{(n-2)(n-3)}{n(n-1)}Bx^{n-4} - \frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)}Cx^{n-5} + \&c. = 0$$
$$x^{n-3} - \frac{n-3}{n}Ax^{n-4} + \frac{(n-3)(n-4)}{n(n-1)}Bx^{n-5} - \frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)}Cx^{n-6} + \&c. = 0$$
$$x^{n-4} - \frac{n-4}{n}Ax^{n-5} + \frac{(n-4)(n-5)}{n(n-1)}Bx^{n-6} - \frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}Cx^{n-7} + \&c. = 0$$
&c.

317. Hoc igitur modo iudicium ad aequationem dati gradus inferioris, quam eſt ipſa propoſita, reduci poteſt. Sic ſi m fuerit numerus quicunque minor quam n, tum ſi aequatio propoſita omnes radices habeat reales, tum quoque huius aequationis gradus m omnes radices erunt reales:

$$x^m - \frac{m}{n}Ax^{m-1} + \frac{m(m-1)}{n(n-1)}Bx^{m-2} - \frac{m(m-1)(m-2)}{n(n-1)(n-2)}Cx^{m-3} + \&c. = 0.$$

Quare ſi ponatur $m = 2$, prodibit iſta aequatio:

$$x^2 - \frac{2}{n}Ax + \frac{2.1}{n(n-1)}B = 0,$$

cuius radices debebunt eſſe reales, ſi quidem aequatio propoſita $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$, omnes habeat radices reales. Cum autem iſta aequatio quadratica radices reales habere nequeat, niſi ſit $\frac{AA}{nn} > \frac{2.1}{n(n-1)}B$, ſequitur, aequationis propoſitae radices omnes reales eſſe non poſſe, niſi ſit $AA > \frac{2n}{n-1}B$. Quamobrem ſi fuerit $AA < \frac{2n}{n-1}B$, hoc

certum erit ſignum, aequationis propoſitae ad minimum duas radices fore imaginarias.

318. Hinc ergo aſſecuti ſumus affectionem neceſſariam, qua coefficientes trium primorum terminorum affecti eſſe debent, ſi quidem aequationis propoſitae omnes radices fuerint reales. Hocque eſt eiuſmodi criterium, uti initio meminimus: ſcilicet etiamſi caſu $AA > \frac{2n}{n-1}B$, nihil pro realitate radicum ſequatur, at ſi ſit $AA < \frac{2n}{n-1}B$, hoc tamen certum ſit ſignum duarum ſaltem radicum imaginariarum. Sic ut omnes radices ſint reales, ſucceſſive pro n numero 2, 3, 4, 5, &c. ſubſtituendo requiritur, ut ſequitur:

$x^2 - Ax + B = 0$ . . . . $A^2 > 4B$

$x^3 - Ax^2 + Bx - C = 0$ . . . . . . $A^2 > \frac{6}{2}B$

$x^4 - Ax^3 + Bx^2 - Cx + D = 0$ . . . $A^2 > \frac{8}{3}B$

$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$ . . $A^2 > \frac{10}{4}B$

Hinc ſi terminus ſecundus deſit, tertiique coefficiens B ſit affirmativus, ut aequatio ſit huiuſmodi:

$$x^n + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0,$$

haec omnes radices reales habere nequit, ſed ad minimum duae erunt imaginariae.

319. Huiuſmodi vero criteria pro coefficientibus ſequentium terminorum erui poſſunt, ſi perpendamus aequationem hanc:

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \&c. = 0$$

totidem habere radices tam reales quam imaginarias, quot ipſa aequatio propoſita contineat. Haec enim aequatio ex illa oritur, ſi ponatur $x = \frac{1}{y}$, ita ut ex radicibus huius aequationis ſimul radices illius habeantur. Quare ſi aequatio propoſita omnes radices habeat reales, tum quoque reciproce iſtius differentialis, ſcilicet huius

— A

$$-A+2By-3Cy^2+4Dy^3-\&c.=0$$

radices omnes erunt reales. Subſtituatur in hac iterum x pro $\frac{1}{y}$, atque emerget iſta aequatio:

$$Ax^{n-1}-2Bx^{n-2}+3Cx^{n-3}-4Dx^{n-4}+\&c.=0,$$

cuius radices propterea omnes erunt reales, ſi radices aequationis propoſitae fuerint tales. Hinc iam patet, ſi fuerit $n=3$, neceſſe eſſe ut ſit $BB>3AC$.

320. Differentietur autem iſta aequatio ulterius atque prodibunt

$$Ax^{n-2}-\frac{2(n-2)}{n-1}Bx^{n-3}+\frac{3(n-2)(n-3)}{(n-1)(n-2)}Cx^{n-4}-\&c.=0$$

$$Ax^{n-3}-\frac{2(n-3)}{n-1}Bx^{n-4}+\frac{3(n-3)(n-4)}{(n-1)(n-2)}Cx^{n-5}-\&c.=0$$

$$Ax^{n-4}-\frac{2(n-4)}{n-1}Bx^{n-5}+\frac{3(n-4)(n-5)}{(n-1)(n-2)}Cx^{n-6}-\&c.=0$$

&c.

Generaliter ergo, ſi m ſit numerus minor quam n, erit:

$$Ax^m-\frac{2m}{n-1}Bx^{m-1}+\frac{3m(m-1)}{(n-1)(n-2)}Cx^{m-2}-\&c.=0.$$

Si iam ponatur $m=2$, habebitur iſta aequatio:

$$Ax^2-\frac{4}{n-1}Bx+\frac{6}{(n-1)(n-2)}C=0,$$

cuius radices ut ſint reales, oportet eſſe $\frac{4BB}{(n-1)^2}>\frac{6AC}{(n-1)(n-2)}$. Quare ſi aequatio propoſita omnes habeat radices reales erit $BB>\frac{3(n-1)}{2(n-2)}AC$. Atque ſi fuerit $BB<\frac{3(n-1)}{2(n-2)}AC$, hoc certum eſt ſignum, aequationem propoſitam ad minimum duas habere radices imaginarias. Si igitur ſit $n=3$, criterium erit $BB>3AC$; ſi ſit $n=4$; erit $BB>\frac{3\cdot3}{2.2}AC$; ſi $n=5$,

$n = 5$, erit $BB > \frac{3 \cdot 4}{2 \cdot 3} AC$, & ita porro.

321. Ut haec criteria ad ſequentes coëfficientes transferamus, reſumamus aequationem differentialem in y inventam

$$-A + 2By - 3Cy^2 + 4Dy^3 - 5Ey^4 + \&c. = 0$$

hancque denuo differentiemus, ut habeamus:

$$2B - 6Cy + 12Dy^2 - 20Ey^3 + \&c. = 0$$

quae reſtituto $\frac{1}{x}$ loco y dabit:

$$Bx^{n-2} - 3Cx^{n-3} + 6Dx^{n-4} - 10Ex^{n-5} + \&c. = 0$$

ex cuius ulteriori differentiatione ſequuntur hae aequationes:

$$Bx^{n-3} - \frac{3(n-3)}{n-2} Cx^{n-4} + \frac{6(n-3)(n-4)}{(n-2)(n-3)} Dx^{n-5} - \&c. = 0$$

& generaliter

$$Bx^m - \frac{3m}{n-2} Cx^{m-1} + \frac{6m(m-1)}{(n-2)(n-3)} Dx^{m-2} - \&c. = 0$$

Quodſi igitur ponamus $m = 2$ prodibit aequatio quadrata:

$$Bx^2 - \frac{2 \cdot 3}{n-2} Cx + \frac{6 \cdot 2}{(n-2)(n-3)} D = 0$$

cuius radices erunt reales, ſi fuerit $\frac{9CC}{(n-2)^2} > \frac{6 \cdot 2BD}{(n-2)(n-3)}$ ſeu $CC > \frac{4(n-2)}{3(n-3)} BD$. Quare ſi aequatio propoſita omnes radices habeat reales, erit $CC > \frac{4(n-2)}{3(n-3)} BD$, atque ſi haec conditio deficiat, aequatio certo duas ad minimum habebit radices imaginarias.

322. Si aequationem ſuperiorem $2B - 6Cy + 12Dy^2 - \&c. = 0$ denuo differentiemus, prodibit:

$$-6C + 24Dy - 60Ey^2 + \&c. = 0, \text{ ſive}$$

$$C - 4Dy + 10Ey^2 - 20Fy^3 + \&c. = 0,$$

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quae reſtituto x loco $\frac{1}{y}$ abibit in hanc:

$$Cx^{n-3} - 4Dx^{n-4} + 10Ex^{n-5} - 20Fx^{n-6} + \&c. = 0$$

ex cuius ulteriori differentiatione ſequuntur:

$$Cx^{n-4} - \frac{4(n-4)}{n-3}Dx^{n-5} + \frac{10(n-4)(n-5)}{(n-3)(n-4)}Ex^{n-6} - \&c. = 0$$

$$Cx^{n-5} - \frac{4(n-5)}{n-3}Dx^{n-6} + \frac{10(n-5)(n-6)}{(n-3)(n-4)}Ex^{n-7} - \&c.$$

& generaliter

$$Cx^{m} - \frac{4m}{n-3}Dx^{m-1} + \frac{10m(m-1)}{(n-3)(n-4)}Ex^{m-2} - \&c. = 0.$$

Ponamus $m = 2$, eritque $Cx^2 - \frac{2.\ 4}{n-3}Dx + \frac{2.\ 10}{(n-3)(n-4)}E = 0$ ex qua ſi eius radices ſint reales ſequitur fore:

$$\frac{4.\ 4}{(n-3)^2}DD > \frac{2.\ 10}{(n-3)(n-4)}CE \text{ ſeu } DD > \frac{5(n-3)}{4(n-4)}CE.$$

323. Ex his iam ſatis perſpicitur relatio omnium coefficientium. Generatim ergo ſi aequatio haec:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \&c. = 0$$

omnes radices habeat reales; erit

$$AA > \frac{2n}{1(n-1)}B$$

$$BB > \frac{3(n-1)}{2(n-2)}AC$$

$$CC > \frac{4(n-2)}{3(n-3)}BD$$

$$DD > \frac{5(n-3)}{4(n-4)}CE$$

$$EE > \frac{6(n-4)}{5(n-5)}DF$$

&c.

Qua-

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Quarum conditionum ſi una deſit, aequatio ad minimum duas habebit radices imaginarias. Atque ſi iſta criteria a ſe invicem non pendeant, facile perſpicitur, quotquot eorum non conveniant, totidem dari paria radicum imaginariarum. Quamvis autem hae conditiones omnes in quapiam aequatione locum habeant, tamen inde non ſequitur, nullas dari radices imaginarias; quin potius evenire poteſt, ut hoc non obſtante omnes radices ſint imaginariae. Cavendum ergo eſt, ne his criteriis plus tribuatur, quam ipſis vi principiorum, unde ſunt deducta, tribui poteſt.

324. Facile autem apparet non ſingula criteria, quae deficiunt, binas radices imaginarias indicare poſſe; in aequatione enim n dimenſionum, quia habentur $n+1$ termini, atque ex ſingulis praeter primum & ultimum criterium deſumi poteſt, omnino criteria habebuntur $n-1$; neque tamen ſi ſingula deficiant, aequatio $2n-2$ radices imaginarias habere poterit, propterea quod omnino tantum n habeat radices. Unum autem criterium ſemper duas radices imaginarias patefacit, & quia fieri poteſt, ut duo criteria huiuſmodi radicum non plures oſtendant, videndum eſt utrum haec duo criteria ſint contigua nec ne: priori caſu numerus radicum imaginariarum non augebitur, poſteriori vero, quia criteria litteras prorſus diverſas involvunt, unumquodque binas radices imaginarias monſtrabit. Ita etiamſi fuerit

$$AA < \frac{2n}{1(n-1)}B \ \& \ BB < \frac{3(n-1)}{2(n-2)}AB,$$ tamen hinc non neceſſario quatuor radices imaginariae indicantur, ſed utrumque fortaſſe eaſdem binas indicat. Quodſi vero fuerit

$$AA < \frac{2n}{1(n-1)}B \ \& \ CC < \frac{4(n-2)}{3(n-3)}BD,$$ exiſtente $$BB > \frac{3(n-1)}{2(n-2)}AC,$$ quatuor radices imaginariae indicabuntur.

325. Ex criteriis ergo radicum imaginariarum ſe immediate

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inſequentibus plus non ſequitur quam ex uno; ſin autem ea ordine interrupto procedant, ut inter bina quaeque criterium unum vel plura contraria interiaceant, tum ex unoquoque binae radices imaginariae concludi poterunt. Quae conſideratio ſequentem regulam ſuppeditat. Aequationis propoſitae ſingulis terminis, praeter primum & ultimum, inſcribantur coefficientes criteriorum ante inventi, hoc modo:

$$\frac{2n}{1(n-1)} \quad \frac{3(n-1)}{2(n-2)} \quad \frac{4(n-2)}{3(n-3)} \quad \frac{5(n-3)}{4(n-4)} \quad \&c.$$

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$$

$$+ \quad .. \quad . \quad .. \quad .. \quad .. \quad \&c.$$

Tum examinetur quadratum cuiuſque coefficientis, utrum ſit maius an minus, quam fractio inſcripta per productum adiacentium coefficientium multiplicata, priori caſu termino ſubſcribatur ſignum +, poſteriori ſignum —; primo vero termino & ultimo perpetuo ſignum + ſubſcribatur. Quo facto, quot ſignorum horum ſubſcriptorum variationes occurrunt, totidem radices imaginarias aequatio ad minimum habere cenſenda erit.

326. Haec eſt regula a *Newtono* inventa ad radices imaginarias cuiuſque aequationis explorandas; de qua autem probe tenendum eſt, quod iam annotavimus, ſaepenumero fieri poſſe, ut aequatio plures habeat radices imaginarias, quam hac methodo deteguntur. Hinc alii operam dederunt, ut ſimiles regulas alias invenirent, quae numerum radicum imaginariarum exactius praeberent, ita ut verus iſtiuſmodi radicum numerus minus ſaepe eum, quem regula oſtendat, excederet. In hoc genere imprimis proſtat regula *Campbelli* Arithmeticae *Newtoni* univerſali ſubiuncta, quam propterea hic explicari conveniet, etiamſi non ſit perfecta. Nititur autem hoc lemmate: Si fuerint α, β, γ, δ, ε, &c. quantitates; earumque numerus ſit m, ponatur ſumma harum quantitatum $\alpha + \beta + \gamma + \delta +$ &c. $= S$, ſumma quadratorum $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 +$ &c. $= V$, erit utique $V > 0$.

Sed cum ſit productum ex binis $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \&c.$ $= \frac{SS - V}{2}$; erit $(m-1)\,V > SS - V$ ſeu $mV > SS$. Nam ſi differentiarum inter binas quantitates quadrata ſumantur, erit eorum ſumma

$$= (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\alpha-\delta)^2 + (\beta-\gamma)^2 + (\beta-\delta)^2 + \&c.$$
$$= (m-1)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.) - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \&c.)$$
$$= (m-1)\,V - 2\,\frac{SS - V}{2} = mV - SS.$$

Cum igitur ſumma quadratorum realium ſit ſemper affirmativa, erit $mV - SS > 0$ ideoque $mV > SS$.

327. Hoc lemmate praemiſſo ſi habeatur haec aequatio:
$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - \&c. = 0,$$
eiuſque omnes radices fuerint reales numero n, quae ſint a, b, c, d, e, &c. erit uti conſtat ex natura aequationum:

	numerus termin.
$A = a + b + c + d + \&c.$	n
$B = ab + ac + ad + bc + bd + \&c.$	$\frac{n(n-1)}{1 \cdot 2}$
$C = abc + abd + abe + acd + bcd + \&c.$	$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$
$D = abcd + abce + abde + \&c.$	$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$

&c.

Sumantur iam ſingulorum harum ſerierum terminorum quadrata, ac ponatur:

$$P = a^2 + b^2 + c^2 + d^2 + \&c.$$
$$Q = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + \&c.$$
$$R = a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + \&c.$$
$$S = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + \&c. \qquad \&c.$$

erit ex natura combinationum:

$$P = A^2 - 2B$$
$$Q = B^2 - 2AC + 2D$$

R =

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$R = C^2 - 2BD + 2AE - 2F$
$S = D^2 - 2CE + 2BF - 2AG + 2H$ &c.

328. Vi igitur lemmatis praemiſſi habebimus:

$$n\,P > AA$$

$$\frac{n(n-1)}{1.\;2}Q > BB$$

$$\frac{n(n-1)(n-2)}{1.\;2.\;3}R > CC$$

$$\frac{n(-1)(n-2)(n-3)}{1.\;2.\;3.\;4}S > DD \quad \&c.$$

Quodſi ergo loco P, Q, R, &c. valores ante inventi ſubſtituantur, obtinebimus ſequentes radicum realium proprietates:

$$nAA - 2nB > AA \text{ ſeu } AA > \frac{2n}{n-1}B$$

$$\frac{n(n-1)}{1.\;2}BB - \frac{2n(n-1)}{1.\;2}AC + \frac{2n(n-1)}{1.\;2}D > BB, \text{ ſive}$$

$$BB > \frac{\frac{2n(n-1)}{1.\;2}}{\frac{n(n-1)}{1.\;2.} - 1}(AC - D)$$

ſimilique modo aequationes ſequentes praebent:

$$CC > \frac{\frac{2n(n-1)(n-2)}{1.\;2.\;3}}{\frac{n(n-1)(n-2)}{1.\;2.\;3} - 1}(BD - AE + F)$$

$$DD > \frac{\frac{2n(n-1)(n-2)(n-3)}{1.\;2.\;3.\;4}}{\frac{n(n-1)(n-2)(n-3)}{1.\;2.\;3.\;4} - 1}(CE - BF + AG - H)$$

Hinc ergo cuiuſque coefficientis quadratum non ſolum cum pro-

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producto proxime adiacentium comparatur, ſed etiam cum rectangulis binorum quorumque utrinque aeque diſtantium; ita tamen ut horum rectangulorum ſigna alternatim mutentur.

329. Singulis igitur aequationis terminis praeter primum & ultimum inſcribi debent fractiones, quarum numeratores ſint unciae binomii ad ſimilem dignitatem elevati duplicatae, denominatores vero eaedem unciae unitate minutae. Ita conſiderando aequationes quadratas, cubicas, biquadratas &c. ſi earum radices omnes fuerint reales, erit:

$$x^2 - \overset{\frac{4}{1}}{A}x + B = 0 \quad ; \; A^2 > 4B$$

Pro aequatione cubica:

$$x^3 - \overset{\frac{6}{2}}{A}x^2 + \overset{\frac{6}{2}}{B}x - C = 0$$

erit $A^2 > 3B$ & $B^2 > 3AC$.

Pro aequatione biquadrata:

$$x^4 - \overset{\frac{8}{3}}{A}x^3 + \overset{\frac{12}{5}}{B}x^2 - \overset{\frac{8}{3}}{C}x + D = 0$$

erit $A^2 > \frac{8}{3}B \; ; \; B^2 > \frac{12}{5}(AC - D) \; ; \; C^2 > \frac{8}{3}BD$

Pro aequatione poteſtatis quintae:

$$x^5 - \overset{\frac{10}{4}}{A}x^4 + \overset{\frac{20}{9}}{B}x^3 - \overset{\frac{20}{9}}{C}x^2 + \overset{\frac{10}{4}}{D}x - E = 0$$

erit $AA > \frac{10}{4}B \; ; \; B^2 > \frac{20}{9}(AC - D) \; ; \; C^2 > \frac{20}{9}(BD - AE)$ & $D^2 > \frac{10}{9}CE$.

Pro aequatione poteſtatis ſextae:

$$x^6 - \overset{\frac{12}{5}}{A}x^5 + \overset{\frac{30}{14}}{B}x^4 - \overset{\frac{40}{19}}{C}x^3 + \overset{\frac{30}{14}}{D}x^2 - \overset{\frac{12}{5}}{E}x + F = 0$$

erit $A^2 > \frac{12}{5}B$; $B^2 > \frac{30}{14}(AC - D)$; $C^2 > \frac{40}{19}(BD - AE + F)$; $D^2 > \frac{30}{14}(CE - BF)$; $E^2 > \frac{12}{5}DF$; &c.

330. Si igitur quodpiam criterium fallat, id erit indicium duas ad minimum ineſſe radices imaginarias in aequatione propoſita. Cum autem ſi ſingula fallant, aequatio ideo non

non duplo plures habere queat radices imaginarias, ſimili modo iudicium his caſibus erit abſolvendum, quem ante pro Neutoniana regula indicavimus. Scilicet ſi cuiuſque termini quadratum maius fuerit quam fractio inſcripta per producta terminorum adiacentium & utrinque aequidiſtantium multiplicata, tum iſti termino ſubſcribatur ſignum +, contra vero ſignum —; primo vero & ultimo termino conſtanter ſubſcribatur ſignum +. Quo facto inſpiciatur ordo ſignorum horum ſubſcriptorum, & quoties occurrit variatio, toties radix imaginaria indicabitur. Quoties ergo haec regula plures radices imaginarias indicat, quam Neutoniana, toties quoque ad veritatem magis accedit. Interim tamen fieri poteſt, ut aequatio plures habeat radices imaginarias, quam per utramque regulam indicantur.

331. Falleremur ergo, ſi his criteriis tanquam perfectis ſignis radicum realium & imaginariarum uti vellemus, propterea quod fieri poteſt, ut aequatio plures habeat radices imaginarias, quam haec criteria indicant: error autem eo maior eſſe poſſet, quo altioris gradus fuerit aequatio propoſita. Nam in aequatione quadrata haec criteria ita veritati ſunt conſentanea, ut ſi nullas radices imaginarias indicent, etiam aequatio nullas ſit habitura. Aequatio autem cubica duas radices imaginarias habere poteſt, etiamſi neutro regula, (ambae autem hoc caſu adhuc conveniunt) eas exhibeat. Hos igitur caſus inveſtigaturi, ſit propoſita haec aequatio cubica generalis:

$$x^3 - \overset{3}{A}x^2 + \overset{3}{B}x - C = 0$$

in qua ſi fuerit $AA > 3B$ & $BB > 3AC$ neutra regula radices imaginarias indicat. Supra autem (306) vidimus ad id, ut nullae radices imaginariae adſint requiri primo ut ſit $B > \frac{1}{3}AA$, quam conditionem quoque ambae regulae requirunt. Sit igitur $B = \frac{1}{3}AA - \frac{1}{3}ff$, atque neceſſe eſt ut C contineatur intra hos limites: $\frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$ & $\frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$. Utraque autem regula tantum poſtulat, ut

ut ſit $C < \frac{BB}{3A}$, hoc eſt $C < \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A}$. Quae conditio locum habere poteſt, etiamſi C non intra dictos limites contineatur.

332. Sit enim $C = \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg$: atque regulae nullas radices imaginarias indicabunt. Interim tamen inerunt duae radices imaginariae, ſi fuerit vel

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 \text{ vel}$$

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg > \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$$

Si igitur fuerit vel $gg > \frac{(ff + Af)^2}{27A}$ vel $gg < \frac{(Af - ff)^2}{27A}$, aequatio cubica duas habebit radices imaginarias, etiamſi neutra regula eas indicet. Sumimus autem hic eſſe A quantitatem affirmativam, ſi enim eſſet negativa, ponendo $x = -y$ aequatio in eiuſmodi formam tranſmutaretur, in qua A eſſet affirmativa. Hinc infinitae aequationes cubicae formari poſſunt, quae habeant duas radices imaginarias, etiamſi per regulam non indicentur. Sit enim $gg = \frac{(ff + Af)^2}{27A} + bb$, erit

$$C = \frac{(AA - ff)^2}{27A} - gg = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - bb, \text{ \& } B = \frac{1}{3}AA - \frac{1}{3}ff.$$

Vel ſit $gg = \frac{(Af - ff)^2}{27A} - bb$ exiſtente $bb < \frac{(Af - ff)^2}{27A}$; erit $C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 + bb$ & $B = \frac{1}{3}AA - \frac{1}{3}ff$. Utroque caſu prodibit aequatio duas habens radices imaginarias, neutra regula indicandas. Ponamus verbi gratia $A = 4$, $f = 1$, erit $B = 5$; & ob $gg = \frac{25}{108} + bb$; erit $C = \frac{225}{108} - \frac{25}{108} - bb = \frac{50}{27} - bb$. Quare ſi ſit $C < \frac{50}{27}$; aequatio $x^3 - 4x^2 + 5x - C = 0$ ſemper habebit duas radices imaginarias. At ſumto $gg = \frac{1}{12} - bb$

debebit eſſe $bb < \frac{1}{12}$, fietque $C = \frac{25}{12} - \frac{1}{12} + bb = 2 + bb$. Sit $bb = \frac{1}{16}$; atque aequatio $x^3 - 4xx + 5x - \frac{33}{16} = 0$ duas habebit radices imaginarias, etiamſi nulla regulis prodatur:

333. Quin etiam eiuſmodi aequationes generales formari poſſunt, in quibus neutra regula radices imaginarias exhibeat, etiamſi tamen ſaepiſſime duae pluresve inſint. Evenit hoc ſi perpetuo duo ſigna ſimilia ſe mutuo excipiant, uti: $x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} - Ex^{n-5} - Fx^{n-6} + \&c. = 0$ vel $x^n + Ax^{n-1} - Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - \&c. = 0$, hic utraque regula nullam unquam radicem imaginariam prodit. Quod autem ſaepiſſime huiuſmodi radices continere queant, vel ex aequatione cubica elucet $x^3 - Ax^2 - Bx + C = 0$, quae poſito $ff = AA + 3B$ ſemper habeat duas radices imaginarias, ſi fuerit vel $-C < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$ vel $-C > \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$. Interim tamen & hos caſus ex regulis elicere licet, ſi aequatio ope ſubſtitutionis in aliam formam transformetur. Ponatur $x = y + k$, fietque:

$$\left.\begin{array}{l} y^3 + 3ky^2 + 3kky + k^3 \\ \quad - Ayy - 2Aky - Akk \\ \qquad\quad - By - Bk \\ \qquad\qquad + C \end{array}\right\} = 0$$

quae ſecundum regulas examinata dabit primo quidem ſponte $(3k - A)^2 > 3(3kk - 2Ak - B)$; at quo ſit $(3kk - 2Ak - B)^2 > 3(3k - A)(k^3 - Akk - Bk + C)$, quod eſt alterum criterium, neceſſe eſt ut ſit:

$$BB + 3AC + (AB - 9C)k + (AA + 3B)kk > 0,$$

quicunque valor ipſi k tribuatur. Sumatur ergo k ita, ut haec expreſſio minimum valorem adipiſcatur, quod fiet ponendo $k = \frac{9C - AB}{2(AA + 3B)}$ & ſi iſta expreſſio adhuc fuerit > 0, probabile erit aequationem propoſitam nullas habere radices imaginarias. Fiet autem $BB + 3AC - \frac{(AB - 9C)^2}{(AA + 3B)} + \frac{(AB - 9C)^2}{4(AA + 3B)} > 0$

ſeu

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ſeu $BB + 3AC > \frac{(AB-9C)^2}{4(AA+3B)}$. Cum ergo ſit $B = \frac{1}{3}ff - \frac{1}{3}AA$, erit $4ff(\frac{1}{9}f^4 - \frac{2}{9}AAff + \frac{1}{9}A^4 + 3AC) > (\frac{1}{3}Aff - \frac{1}{3}A^3 - 9C)^2$ ſeu
$4f^6 - 8A^2f^4 + 4A^4ff + 108ACff > A^2f^4 - 2A^4f^2 - 54ACff + A^6 + 54A^3C + 729CC$
vel $4f^6 > 9A^2f^4 - 6A^4ff - 162ACff + A^6 + 54A^3C + 729CC$
unde factoribus ſumtis eſſe debebit:

$$(2f^3 + A^3 - 3Aff + 27C)(2f^3 - A^3 + 3Aff - 27C) > 0.$$

Hincque regulae radices imaginarias oſtendent, ſi fuerit vel
$C > -\frac{1}{27}A^3 + \frac{1}{9}Aff - \frac{2}{27}f^3$ & $C > -\frac{1}{27}A^3 + \frac{1}{9}Aff + \frac{2}{27}f^3$ vel
$C < -\frac{1}{27}A^3 + \frac{1}{9}Aff - \frac{2}{27}f^3$ & $C < -\frac{1}{27}A^3 + \frac{1}{9}Aff + \frac{2}{27}f^3$.
Quae ſunt eaedem conditiones, quas ſupra invenimus. Patet ergo idonea aequationis propoſitae tranſmutatione regulas hoc capite traditas ita perfici poſſe, ut a veritate non diſſideant, etiamſi convertantur.

334. Ex his principiis quoque regula *Harriotti*, qua quaelibet aequatio tot radices affirmativas habere praedicatur, quot dentur ſignorum variationes, tot vero negativas, quot dentur eiuſdem ſigni ſucceſſiones, demonſtrari poteſt, quae quidem regula pro radicibus tantum realibus valet. Ponamus ergo aequationem $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} -$ &c. $= 0$ omnes radices habere reales atque affirmativas, atque eius differentialis $nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} -$ &c. $= 0$ non ſolum omnes ſuas radices quoque habebit reales & affirmativas, ſed etiam huius radices conſtituent limites radicum illius aequationis. Praeterea vero poſito $x = \frac{1}{y}$ haec aequatio $1 - Ay + By^2 - Cy^3 + Dy^4 -$ &c. $= 0$ omnes quoque radices habebit reales affirmativas, ſed reciprocas illius, ita ut quae radices in illa aequatione ſint maximae, hae in iſta fiant minimae. His poſitis ſi illa aequatio propoſita continuo differentietur, donec ad aequationem primi ordinis perveniatur, quae erit $x - \frac{1}{n}A = 0$, (317) huius radix adhuc

erit affirmativa, ideoque coefficiens ſecundi termini habebit ſignum — uti aſſumſimus. Sin autem iſte coefficiens haberet ſignum +, tum certo ſequeretur, aequationem propoſitam non omnes radices habere affirmativas, ſed unam ad minimum fore negativam, & quidem eam, quae limitibus hucuſque perductis reſpondeat.

335. Si aequatio propoſita in ſuî reciprocam convertatur & differentietur, tum vero iterum x reſtituatur atque differentiationes continuentur, donec perveniatur ad aequationem ſimplicem, quae ex §. 320. erit huiuſmodi $Ax - \frac{2}{n-1}B = o$, cuius propterea radix quoque debet eſſe affirmativa, ſi quidem propoſita omnes ſuas radices habeat reales affirmativas, hincque ſecundus & tertius terminus diverſa ſigna habebunt. Quodſi ergo hi duo termini ſimilia habeant ſigna, ad minimum una radix negativa indicabitur, reſpondens limiti hac aequatione ſignato, qui diverſus erit a limite praecedente aequatione indicato, propterea quod hic radices ſemel ſunt in ſuas reciprocas converſae: unde concluditur, ſi tres termini aequationis initiales paria habuerint ſigna, tum duas radices negativas indicari.

336. Simili modo ſi converſiones & differentiationes ſecundum §. 321. inſtituantur, atque eouſque continuentur, donec ad aequationem ſimplicem $Bx - \frac{3}{n-2}C = o$ perveniatur, & huius aequationis radix eſſe debet affirmativa, ſi quidem propoſitae aequationis omnes radices fuerint tales; unde ſi termini tertius & quartus paria habeant ſigna, indicabitur una radix negativa. Sicque perpetuo, ſi duo quicunque termini contigui aequalibus ſignis fuerint affecti, una radix negativa proditur; ideoque quotcunque fuerint eiuſdem ſigni ſucceſſiones, totidem ad minimum aequatio propoſita habebit radices negativas, quoniam haec ſingula criteria ad diverſos limites referuntur. Quod ſi autem aequatio propoſita omnes radices

ces negativas habere ponatur, tum quia radices omnium aequationum differentialium ex ea deductarum debent eſſe pariter negativae, omnes termini aequalibus ſignis affecti eſſe debebunt. Quare ſi duo termini contigui diverſa habeant ſigna, ex iis una ad minimum radix affirmativa concludetur. Atque ſimili modo, quotcunque in aequatione occurrant binorum terminorum variationes ſignorum, totidem ad minimum radices affirmativae ineſſe dicendae ſunt. Cum igitur aequatio omnis tot habeat radices, quot dantur duorum ſignorum contiguorum combinationes, neque plures, ſequitur quamvis aequationem, cuius omnes radices ſint reales, tot habere radices affirmativas, quot fuerint ſignorum contiguorum variationes, tot vero negativas, quot fuerint eiuſdem ſigni ſucceſſiones.

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CAPUT XIV.

DE DIFFERENTIALIBUS FUNCTIONUM IN CERTIS TANTUM CASIBUS.

337.

Si y fuerit functio quaecunque ipſius x, atque haec quantitas variabilis x augeatur incremento ω, ut x abeat in $x+\omega$, tum functio y induet hunc valorem:

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\&c.$$

ideoque capiet hoc incrementum:

$$\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\&c.$$

uti ſupra demonſtravimus. Quare ſi fiat $\omega=dx$, ita ut x ſuo differentiali dx creſcat, tum functio y incrementum accipet $=dy+\frac{1}{2}ddy+\frac{1}{6}d^3y+\frac{1}{24}d^4y+\&c.$ quod erit verum differentiale ipſius y. Quoniam vero huius ſeriei quilibet terminus ad ſequentes habet rationem infinitam, prae primo omnes evaneſcunt, ita ut dy more conſueto ſumtum praebeat verum differentiale ipſius y. Simili modo vera differentialia ſecunda, tertia, quarta, &c. ipſius y ita ſe habebunt:

$$dd.y=ddy+\frac{3}{3}d^3y+\frac{7}{3.4}d^4y+\frac{15}{3.4.5}d^5y+\frac{31}{3.4.5.6}d^6y+\&c.$$

$$d^3.y=d^3y+\frac{6}{4}d^4y+\frac{25}{4.5}d^5y+\frac{90}{4.5.6}d^6y+\frac{301}{4.5.6.7}d^7y+\&c.$$

$$d^4.y=d^4y+\frac{10}{5}d^5y+\frac{65}{5.6}d^6y+\frac{350}{5.6.7}d^7y+\&c.$$

$$d^5.y=d^5y+\frac{15}{6}d^6y+\frac{140}{6.7}d^7y+\&c.$$

$$d^6.y=d^6y+\frac{21}{7}d^7y+\&c.$$

quae

quae ſequuntur ex §. 56. ſi loco ω ponatur dx. Erunt ergo haec differentialia ipſius y completa, quippe in quibus ne ii quidem termini, qui reſpectu primi evaneſcunt, negliguntur. Inveniuntur autem ſinguli iſti termini, ſi functio y continuo differentietur, ponendo dx conſtans. Sic poſito $y = ax - xx$ ob $dy = adx - 2xdx$ & $ddy = -2dx^2$; erunt ipſius y differentialia completa: $dy = adx - 2xdx - dx^2$; $ddy = -2dx^2$; ſequentia autem ſunt nulla.

338. Quanquam autem generatim in his expreſſionibus differentialium ſequentes termini prae primis pro nihilo reputantur; tamen in caſibus ſpecialibus, quibus ipſe terminus primus evaneſcit, haec ratio ceſſat, neque terminus ſecundus amplius negligi poterit. Sic in exemplo praecedente etiamſi formulae $y = ax - xx$ differentiale in genere eſt $= (a - 2x)dx$ reiecto termino $-dx^2$, quippe qui eſt infinities minor quam primus $(a - 2x)dx$: hic tamen iſta conditio manifeſto ſubintelligitur, niſi primus terminus per ſe evaneſcat. Quocirca ſi ipſius $y = ax - xx$ quaeratur differentiale, caſu quo $x = \frac{1}{2}a$, tum id dicendum erit eſſe $= -dx^2$; ſcilicet ſi variabilis x differentiali dx creſcat, tum functionis y caſu $x = \frac{1}{2}a$ decrementum erit dx^2. Hoc autem ſolo caſu excepto perpetuo functionis y differentiale erit $= (a - 2x)dx$; niſi enim ſit $x = \frac{1}{2}a$, terminus ſecundus $-dx^2$ prae primo ſemper recte negligitur. Neque vero neglectio termini dx^2 etiam in caſu $x = \frac{1}{2}a$ in errorem inducere poteſt: comparari enim differentialia prima inter ſe ſolent; unde quia $dy = -dx^2$ caſu $x = \frac{1}{2}a$, prae differentialibus primis dx evaneſcit, perinde eſt ſive hoc caſu habeamus $dy = 0$ ſive $dy = -dx^2$.

339. Denotante y functionem quamcunque ipſius x, ſit differentialibus continuis ſumtis:

$$dy = pdx;\ dp = qdx;\ dq = rdx;\ dr = sdx;\ \&c.$$

Hinc ergo differentialia completa, in quibus nihil negligatur, ipſius y erunt:

d.y

$$d\,.\,y = pdx + \tfrac{1}{2}\,qdx^2 + \tfrac{1}{6}\,rdx^3 + \tfrac{1}{24}\,sdx^4 + \&c.$$
$$d^2.y = qdx^2 + rdx^3 + \tfrac{7}{12}\,sdx^4 + \tfrac{1}{4}\,tdx^5 + \&c.$$
$$d^3.y = rdx^3 + \tfrac{3}{2}\,sdx^4 + \tfrac{5}{4}\,tdx^5 + \&c.$$
$$d^4.y = sdx^4 + 2\,tdx^5 + \&c.$$
$$d^5.y = tdx^5 + \&c.$$

Niſi ergo primi termini harum expreſſionum evaneſcant, ii ſoli differentialia ipſius y exhibebunt; ſin autem quopiam caſu primus terminus fiat $= 0$, tum ſequens differentiale quaeſitum exprimet. Atque ſi etiam ſecundus terminus evaneſcat, tum tertius terminus valorem differentialis quaeſiti praebebit, ſin autem & hic evaneſcat, quartus & ita deinceps. Unde intelligitur nullius functionis ipſius x differentiale primum unquam penitus evaneſcere; etiamſi enim fiat $p = 0$, quo caſu vulgo dy evaneſcere cenſetur, tum hoc differentiale per altiorem ipſius dx poteſtatem exprimetur. Uti vel per $\frac{1}{2}qdx^2$, vel ſi etiam ſit $q = 0$, per $\frac{1}{6}rdx^3$; & ita porro.

340. Quanquam autem his caſibus differentiale ipſius y reſpectu aliorum differentialium primorum, quibuſcum comparatur, recte negligitur, atque pro nihilo reputatur; tamen ſaepenumero eius veram expreſſionem noſſe iuvat. Ex completa enim differentialis forma ſtatim perſpici poteſt, quibus caſibus data functio fiat maximum vel minimum. Si enim fuerit:

$$d.y = pdx + \tfrac{1}{2}\,qdx^2 + \tfrac{1}{6}\,rdx^3 + \&c.$$

quo y nanciſcatur maximum minimumve valorem, neceſſe eſt ut ſit $p = 0$; erit ergo hoc caſu $dy = \frac{1}{2}\,qdx^2$, & functio y, ſi loco x ponatur $x \pm dx$, abit in $y + \frac{1}{2}\,qdx^2$, eritque propterea minima, ſi q habeat valorem affirmativum, at maxima ſi q habeat valorem negativum. At ſi ſimul fiat $q = 0$, erit $dy = \frac{1}{6}rdx^3$, & functio y ponendo $x \pm dx$ loco x abibit in $y \pm \frac{1}{6}\,rdx^3$, neque hoc caſu maximum neque minimum prodit; ſin autem fiat & $r = 0$, tum poſito $x \pm dx$ loco x functio y evadet $= y + \frac{1}{24}\,sdx^4$, quae maximum exhibet, ſi s fuerit quantitas negativa, minimum vero, ſi s ſit quantitas affirmativa. Aliae occaſiones, quibus differentialium

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lium completa expreſſio uſum habet, infra occurrent.

341. Ponamus p evaneſcere caſu $x = a$, quod evenit ſi fuerit $p = (x - a)P$. Talis autem valor prodit, ſi fuerit $y = (x - a)^2 P + C$, denotante C quantitatem conſtantem quamcunque. Cum enim ſit $pdx = (x-a)^2 dP + 2(x-a)Pdx$, erit utique $p = 0$, poſito $x = a$. Tum ergo ob $dpdx = qdx^2 = (x - a)^2 ddP + 4(x - a)dPdx + 2Pdx^2$, poſito $x = a$, fiet $qdx^2 = 2Pdx^2$, atque differentiale completum hoc caſu $x = a$, erit $d.y = Pdx^2$, niſi forte & P evaneſcat poſito $x = a$, quos caſus poſtea contemplabor. Praeſens autem caſus generalius hoc modo exhiberi poteſt. Sit $z = (x - a)^2 P + C$, atque y ſit functio quaecunque ipſius z, ita ut fiat $dy = Zdz$, denotante Z functionem quamcunque ipſius $z = (x - a)^2 P + C$. Erit ergo $dz = (x-a)^2 dP + 2(x - a)Pdx$, & $pdx = Z(x - a)^2 dP + 2Z(x - a)Pdx$, quod membrum fit $= 0$ ſi $x = a$; eodemque caſu neglectis terminis, qui continent factorem $x - a$, erit $qdx^2 = 2PZdx^2$, ideoque caſu $x = a$, fiet $dy = PZdx^2$; poſtquam in PZ ubique loco x poſitum fuerit a. Quare ſi fuerit y functio quaecunque ipſius $z = (x - a)^2 P + C$, ita ut ſit $dy = Zdz$, erit caſu $x = a$, differentiale $dy = PZdx^2$. Fiet ergo haec functio y maxima caſu $x = a$, ſi eodem caſu fiat PZ quantitas negativa, minima vero, ſi PZ fiat quantitas affirmativa.

342. Si fuerit $p = (x - a)^2 P$, caſu $x = a$ quoque q evaneſcit, talis autem expreſſio pro p oritur, ſi fuerit $y = (x - a)^3 P + C$. Erit ergo $pdx = (x - a)^3 dP + 3(x - a)^2 Pdx$; $qdx^2 = (x - a)^3 ddP + 6(x - a)^2 dPdx + 6(x - a)Pdx^2$, quorum utrumque membrum caſu $x = a$ evaneſcit; at vero ſequens erit $rdx^3 = (x - a)^3 d^3 P + 9(x - a)^2 ddPdx + 18(x - a)dPdx^2 + 6Pdx^3 = 6Pdx^3$, poſito $x = a$. Quare cum & p & q caſu $x = a$ evaneſcat, fiet $dy = \frac{1}{6} rdx^3 = Pdx^3$. Simili modo ſi ponatur $z = (x - a)^3 P + C$, fueritque y functio quaecunque ipſius z; ita ut ſit $dy = Zdx$, ob $dz = (x-a)^3 dP + 3(x-a)^2 Pdx$, fiet quoque $p = 0$ & $q = 0$, eritque $rdx^3 = 6PZdx^3$; unde

caſu $x=a$, erit $dy=PZdx^3$. Quare iſta functio y, etiamſi caſu $x=a$, fiat $p=0$, tamen neque maximum neqne minimum valorem recipit.

343. Haec differentialia facilius inveniri poſſunt ex ipſa differentialium natura. Cum enim differentiale ipſius y oriatur, ſi y a ſtatu ſequenti proximo ſubtrahatur, qui prodit, ſi loco x ponatur $x+dx$; ponamus caſu primo, quo erat $y=(x-a)^2P+C$, $x+dx$ loco x, eritque $y^I=(x-a+dx)^2P^I+C$, unde fiet $dy=(x-a+dx)^2P^I-(x-a)^2P$. Caſu igitur quo $x=a$, erit $dy=P^Idx^2$, & cum P^I ad P rationem aequalitatis habeat, erit $dy=Pdx^2$. Simili modo ſi fuerit $z=(x-a)^2P+C$; erit $dz=Pdx^2$; quare ſi ſit y functio quaecunque ipſius z, ita ut ſit $dy=Zdz$, erit $dy=PZdx^2$ caſu, quo ponitur $x=a$. Deinde ſi ſit $z=(x-a)^3P+C$, erit $z^I=(x-a+dx)^3P^I+C$, & propterea caſu $x=a$, fiet $z^I-z=dz=Pdx^3$. Hinc ſi fuerit y functio quaecunque ipſius z, atque $dy=Zdz$, erit quoque caſu $x=a$, differentiale $dy=PZdx^3$, ſi quidem in functionibus P & Z loco x ubique ſubſtituatur a. Quoniam vero hoc caſu fit $z=C$, atque Z eſt functio ipſius z, evadet Z quantitas conſtans, talis ſcilicet functio ipſius C, qualis ante erat ipſius z.

344. Si igitur generaliter fuerit $y=(x-a)^nP+C$, quia eſt $y^I=(x-a+dx)^nP^I+C$, caſu $x=a$, fiet $dy=Pdx^n$; unde ſi fuerit $n>1$, hoc differentiale reſpectu aliorum differentialium primorum, quae ipſi dx ſunt homogenea, evaneſcet. Ex praecedentibus ergo manifeſtum eſt, functionem y fieri caſu $x=a$, vel maximam vel minimam, ſi fuerit n numerus par: tum enim ſi poſito $x=a$ fiat P quantitas affirmativa, fiet y minimum, ſin autem P ſit quantitas negativa, fiet y maximum. Hocque ergo modo ratio maximorum & minimorum multo facilius invenitur, quam methodo ſupra expoſita quia non opus eſt ad differentialia altiora progredi. Quod ſi vero ſit $z=(x-a)^nP+C$, atque y fuerit functio quaecunque ipſius z, ut ſit $dy=Zdz$, erit caſu $x=a$

diffe-

differentiale $dy = PZdx^n$. Notandum autem eſt, hic n ſumi pro numero affirmativo ſeu o maiore, ſi enim n eſſet numerus negativus, tum poſito $x = a$, non evaneſceret $(x-a)^n$, uti aſſumſimus, ſed adeo fieret infinite magnum.

345. Iam vidimus hoc pacto differentiale multo expeditius inveniri, quam ope ſeriei, qua ante differentiale completum expreſſimus; ſi enim ſit n numerus integer, tot ſeriei illius termini perluſtrari deberent, quot n contineat unitates. Verum ſi n ſit numerus fractus, tum ſeries iſta nequidem verum differentiale unquam exhibebit. Ponamus enim eſſe $y = (x-a)^{\frac{3}{2}} + a\sqrt{a}$, ſi ſeriem

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rdx^3 + \tfrac{1}{24}sdx^4 + \&c.$$

ſpectemus, fiet $p = \frac{3}{2}\sqrt{(x-a)}$, $q = \dfrac{3}{4\sqrt{(x-a)}}$,

$$r = \frac{-3}{8(x-a)\sqrt{(x-a)}}, \quad s = \frac{9}{16(x-a)^2\sqrt{(x-a)}}, \ \&c.$$

Quare ſi ponatur $x = a$ fiet quidem $p = 0$, at ſequentes termini omnes q, r, s, &c. evadent infiniti; unde valor differentialis dy hoc caſu omnino definiri non poteſt. At vero methodus ex ipſa differentialium natura deducta nullum dubium relinquit. Cum enim ſit $y = (x-a)^{\frac{3}{2}} + a\sqrt{a}$, poſito $x+dx$ loco x fiet $y^{I} = (x-a+dx)^{\frac{3}{2}} + a\sqrt{a}$, eritque, ſi $x = a$ ponatur, $dy = dx\sqrt{dx}$. Evaneſcit ergo hoc differentiale prae dx, at vero differentialia ſecunda cum dx^2 homogenea prae eo evaneſcent.

346. Evolvamus hos caſus, quibus exponens n eſt numerus fractus aliquanto accuratius, ſitque $y = P\sqrt{(x-a)} + C$; ob $y^{I} = P^{I}\sqrt{(x-a+dx)} + C$, fiet $dy = P\sqrt{dx}$ caſu $x = a$; unde hoc differentiale ad dx, & ad differentialia cum dx homogenea rationem tenebit infinitam. Hinc etiam patet, quid hoc caſu de ratione maximi ac minimi ſit tenendum. Cum enim

 poſito

poſito $x + dx$ loco x, abeat y in $P\surd(x-a) + C + P\surd dx$, ob $\surd dx$ ambiguum, functio y geminum induet valorem, alterum maiorem quam C, quem recipit poſito $x = a$, alterum minorem; unde caſu $x = a$ neque maximum neque minimum fiet. Praeterea ſi dx capiatur negative tum valor ipſius y adeo fiet imaginarius. Idem tenendum eſt ſi ſit $z = P\surd(x-a) + C$, & y functio quaecunque ipſius z, ut ſit $dy = Zdz$, tum enim erit $dy = PZ\surd dx$ caſu $x = a$.

347. Si propoſita fuerit iſta functio $y = (x-a)^{\frac{m}{n}}P + C$, cuius differentiale quaeritur caſu $x = a$, erit uti ex antecedentibus colligitur $dy = Pdx^{\frac{m}{n}}$. Quocirca ſi fuerit $m > n$ hoc differentiale prae dx evaneſcet; ſin autem ſit $m < n$, ratio $\frac{dy}{dx}$ erit infinita magna. Praeterea vero ſi n ſit numerus par, differentiale dy geminum habebit valorem, alterum affirmativum, alterum negativum; ſicque functio y, quae caſu $x = a$ fit $= C$, ſi ponatur $x = a + dx$, binos habebit valores alterum maiorem quam C alterum vero minorem; ſin autem poneretur $x = a - dx$, tum y adeo fieret imaginarium; unde hoc caſu y neque maximum fit neque minimum. Ponamus nunc denominatorem n eſſe numerum imparem, erit numerator m vel par vel impar. Sit primo m numerus par; quia dy eundem valorem retinet, ſive dx ſumatur affirmative ſive negative, perſpicuum eſt, functionem y caſu $x = a$ fieri ſive maximam ſive minimam, prout hoc caſu fuerit P vel quantitas negativa vel affirmativa. Sin autem uterque numerus m & n fuerit impar, differentiale dy in ſui negativum abibit, poſito dx negativo; hocque ergo caſu functio y neque maximum erit neque minimum, ſi ponatur $x = a$.

348. Si functio y ex pluribus huiuſmodi terminis, quorum ſinguli ſint diviſibiles per $x - a$, conſtet, ita ut ſit $y = (x-a)^m P + (x-a)^n Q + C$, tum eius differentiale caſu $x = a$ erit $dy = Pdx^m + Qdx^n$; in qua expreſſione,

ſi

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ſi fuerit $n > m$, terminus ſecundus prae primo evaneſcit, ita ut tantum prodeat $dy = Pdx^m$. Sin autem n ſit fractio denominatorem habens parem, tum etiamſi Qdx^n prae Pdx^m evaneſcat, tamen omnino negligi non poteſt. Ex eo enim apparet, ſi capiatur dx negative, valorem ipſius dy fieri imaginarium, quod ex ſolo termino primo Pdx^m non patet. Cum ergo ſi n ſit fractio denominatorem habens parem, dx negative accipi nequeat, ſin autem affirmative capiatur, terminus Qdx^n geminum praebeat valorem: functio $y = (x-a)^m P + (x-a)^n Q + C$ quae caſu $x = a$ fit $= C$, ſi ponatur $x = a + dx$, erit $y = C + Pdx^m \pm Qdx^n$, quorum valorum uterque cum vel maior ſit vel minor quam C, prout P fuerit quantitas vel affirmativa vel negativa, erit functio y caſu $x = a$ vel minimum vel maximum ſecundae ſpeciei.

349. His igitur caſibus differentialia functionum vera non per regulas differentiationis conſuetas inveniri poſſunt; quippe quae tantum valent, quamdiu differentiale functionis eſt homogeneum cum dx. Sin autem caſu quopiam ſingulari differentiale functionis exprimatur per eius poteſtatem dx^n, tum regula praebet pro hoc differentiali o, ſi n fuerit numerus unitate maior; at vero differentiale exhibet infinite magnum, ſi n ſit exponens unitate minor. Sic ſi ipſius $y = \sqrt{(a-x)}$ differentiale quaeratur caſu $x = a$, quia eſt $dy = -\frac{dx}{\sqrt{(a-x)}}$, facto $x = a$ prodit $dy = -\frac{dx}{o}$. Atque ſi differentialia ſequentia in ſubſidium vocare velimus, omnia pariter ob denominatores $= o$ in infinitum excreſcunt, ita ut inde nihil concludi poſſit. At vero hoc caſu vidimus eſſe $dy = \sqrt{-dx}$, atque adeo imaginarium. Sin autem loco x ponatur $x - dx$, erit $dy = \sqrt{dx}$, atque adeo erit infinities maius quam dx, ita ut dx prae dy evaneſcat. Quare regula conſueta etiam hoc caſu in errorem non inducit, cum valorem ipſius dy infinitum exhibeat.

350. A regula ergo conſueta differentiationis recedendum eſt, quoties in ſerie $p\,dx + \frac{1}{2}q\,dx^2 + \frac{1}{6}r\,dx^3 +$ &c. qua differentiale completum functionis y exprimitur, primus terminus p vel fit $= 0$ vel in infinitum excreſcit, eoque caſu differentiale ex primis principiis derivari debet. Quoties ergo functionis y differentiale quaeritur dato ipſius x valori reſpondens, quo littera p vel infinite parva evadit vel infinite magna, toties recurrendum eſt ad ipſa prima differentiationis principia. Omnibus vero reliquis caſibus, quibus fit neque $p = 0$ neque $p = \infty$, conſueta regula veros differentialis valores praebebit. Interim tamen caſus ante (348) memoratus non eſt negligendus, ſi functio y contineat huiuſmodi membrum $(x - a)^n Q$ exiſtente n fractione denominatorem parem habente; etiamſi enim adſint differentialia inferiora quam $Q\,dx^n$, prae quibus hoc evaneſcat; tamen quoniam $Q\,dx^n$ ſi ſit dx negativum, fit imaginarium, hoc membrum $Q\,dx^n$ reliqua omnia, prae quibus evaneſcit, quoque tranſmutat in imaginaria: cuius circumſtantiae ratio potiſſimum in lineis erit habenda. Huiuſmodi ergo caſus particulares, quibus verum differentiale communi regula non indicatur, in adiunctis exemplis explicabo.

EXEMPLUM I.

Quaeratur differentiale functionis

$$y = a + x - \sqrt{[xx + ax - x\sqrt{(2ax - xx)}]}$$

caſu quo ponitur $x = a$.

Differentiale iſtius functionis caſu $x = a$ per regulam receptam non reperiri, ex differentiatione patet, fit enim:

$$dy = dx - \frac{x\,dx - \frac{1}{2}a\,dx + \frac{1}{2}dx\sqrt{(2ax - xx)} + (ax\,dx - xx\,dx) : \sqrt{(2ax - xx)}}{\sqrt{[xx + ax - x\sqrt{(2ax - xx)}]}}$$

poſito enim $x = a$ erit $dy = dx - \frac{a\,dx}{a} = 0$. Ordiamur ergo a principiis differentiationis, ac primo quidem poſito $x + dx$ loco x fiet:

$$y^{I} = a + x + dx - \sqrt{[xx + 2x\,dx + dx^2 + ax + a\,dx - (x + dx)\sqrt{(2ax - xx + 2a\,dx - 2x\,dx - dx^2)}]}$$

Poſi-

Poſito autem $x = a$ erit:

$y' = 2a + dx - \sqrt{[2aa + 3adx + dx^2 - (a + dx)\sqrt{(aa - dx^2)}]}$

iam cum ſit $\sqrt{(aa - dx^2)} = a - \frac{dx^2}{2a}$, ſequentes enim termini tuto negligi poterunt, quia non omnes, qui ſunt infinities maiores, deſtruentur, ut mox patebit:

erit $y' = 2a + dx - \sqrt{(aa + 2adx + \frac{3}{2}dx^2)}$, porroque radicem extrahendo fiet $y' = 2a + dx - \left(a + dx + \frac{dx^2}{4a}\right) = a - \frac{dx^2}{4a}$. At caſu $x = a$, erit $y = a$; unde cum ſit $y' = y + dy$ obtinebitur $dy = -\frac{dx^2}{4a}$: ex quo ſimul perſpicitur functionem propoſitam y fieri maximum, ſi ponatur $x = a$.

EXEMPLUM II.

Invenire differentiale huius functionis:

$$y = 2ax - xx + a\sqrt{(aa - xx)}$$

caſu, quo ponitur $x = a$.

Facta differentiatione more conſueto fit $dy = 2adx - 2xdx - \frac{axdx}{\sqrt{(aa - xx)}}$, quod poſito $x = a$ in infinitum abit, neque ergo hoc modo indicatur. Differentialia vero ſequentium ordinum pariter omnia fient infinita, ita ut ex iis nequidem ex ſerie $pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 +$ &c. verus valor differentialis inveniri queat. Ponamus ergo $x + dx$ loco x, atque habebimus $y' = 2ax - xx + 2adx - 2xdx - dx^2 + a\sqrt{(aa - xx - 2xdx - dx^2)}$ & poſito $x = a$ erit: $y' = aa - dx^2 + a\sqrt{(-2adx - dx^2)}$ At eodem caſu fit $y = aa$; unde erit $dy = -dx^2 + a\sqrt{-2adx}$, & cum dx^2 prae $\sqrt{-2adx}$ evaneſcat, erit $dy = a\sqrt{-2adx}$. Quare ſi differentiale dx affirmative capiatur, erit dy imaginarium; ſin autem pro x ſcribatur $x - dx$, erit $dy = a\sqrt{2adx}$, cuius cum duplex ſit valor alter affirmativus, alter negativus, functio y caſu $x = a$ neque maxima fiet neque minima.

EXEM-

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EXEMPLUM III.

Invenire differentiale functionis:

$$y = 3aax - 3axx + x^3 + (a - x)^2 \sqrt[3]{(a^3 - x^3)}$$

caſu quo ponitur $x = a$.

Quoniam haec functio in iſtam formam transformatur $y = a^3 - (a - x)^3 + (a - x)^{\frac{7}{3}} \sqrt[3]{(aa + ax + xx)}$, poſito $x = a + dx$ fit $y' = a^3 + dx^3 - dx^{\frac{7}{3}} \sqrt[3]{3aa}$, eodemque caſu eſt $y = a^3$. Erit ergo $dy = dx^3 - dx^{\frac{7}{3}} \sqrt[3]{3aa}$, & cum dx^3 evaneſcat prae $dx^{\frac{7}{3}}$, erit $dy = - dx^{\frac{7}{3}} \sqrt[3]{3aa}$. Caſu ergo $x = a$ functio y neque maximum fit neque minimum.

EXEMPLUM IV.

Invenire differentiale functionis:

$$y = \sqrt{x} + \sqrt[4]{x^3} = (1 + \sqrt[4]{x})\sqrt{x} \quad \textit{caſu } x = 0.$$

Quoniam caſus $x = 0$ proponitur, eoque fit $y = 0$, loco x tantum dx ſcribatur, & habebitur $dy = dx^{\frac{1}{2}} + dx^{\frac{3}{4}}$, ſeu $dy = (1 + \sqrt[4]{dx})\sqrt{dx}$; unde primum patet dx negative accipi non poſſe. Tum vero etiamſi alias $\sqrt{dx}$ geminum valorem prae ſe ferat, alterum affirmativum alterum negativum, tamen hoc caſu, quia eius radix $\sqrt[4]{dx}$ occurrit, non niſi affirmative accipi poteſt. At vero $\sqrt[4]{dx}$ utrumque ſignificatum recipit, eritque $dy = \sqrt{dx} \pm \sqrt[4]{dx^3}$ & $y' = 0 + \sqrt{dx} \pm \sqrt[4]{dx^3}$, ob $y = 0$. Cum igitur uterque ipſius y' valor maior ſit, quam ipſius y, ſequitur caſu $x = 0$ fieri y minimum. Quod autem functio $y = \sqrt{x} + \sqrt[4]{x^3}$ non complectatur hanc $y = -\sqrt{x} + \sqrt[4]{x^3}$, utramque ad rationalitatem perducendo patebit. Prior enim fuſa in hanc formam $y - \sqrt{x} = \sqrt[4]{x^3}$, & qua-

quadrata dat, $y^2 - 2y\sqrt{x} + x = x\sqrt{x}$ ſeu $y^2 + x = (x + 2y)\sqrt{x}$, quae denuo quadrata praebet $y^4 - 2y^2x - 4xxy + xx - x^3 = 0$. Altera vero $y + \sqrt{x} = \sqrt[4]{x^3}$ dabit $y^2 + x = (x - 2y)\sqrt{x}$ & porro $y^4 - 2yyx + 4xxy + xx - x^3 = 0$ quae ab illa eſt diverſa. At vero alterum membrum $\sqrt[4]{x^3}$ ambiguitatem ſigni retinet. Quamobrem iſta circumſtantia probe eſt notanda, quod etiamſi communiter radices poteſtatum parium utrumque ſignum $+$ & $-$ includant, tamen haec ambiguitas ceſſet, ſi in eadem expreſſione earumdem radicum ulteriores radices poteſtatum parium occurrant; quippe quae fierent imaginariae, ſi radices priores negative acciperentur. Atque ex hoc fonte maxima & minima ſecundae ſpeciei ſequuntur, quando talia non locum habere videantur.

EXEMPLUM V.

Invenire differentiale functionis:

$$y = a + \sqrt{(x-f)} + (x-f)\sqrt[4]{(x-f)} + (x-f)^2\sqrt[8]{(x-f)}$$

caſu quo ponitur $x = f$.

Ponamus $x - f = t$, & cum ſit $y = a + \sqrt{t} + t\sqrt[4]{t} + tt\sqrt[8]{t}$ huius differentiale quaeritur caſu $t = 0$, quo fit $y = a$. Poſito ergo $t + dt$ ſeu $0 + dt$ loco t fiet $y' = y + dy = a + \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}$, ideoque habebitur $dy = \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}$. Ubi primo patet differentiale dt negative accipi non poſſe, quin dy fiat imaginarium. Tum verum non ſolum $\sqrt{dt}$, ſed nequidem $\sqrt[4]{dt}$ negative accipi poteſt; fieret enim $\sqrt[8]{dt}$ imaginarium: unde differentiale dy geminum tantum habet valorem, $dy = \sqrt{dt} + dt\sqrt[4]{dt} \pm dt^2\sqrt[8]{dt}$, quorum cum uterque maior ſit nihilo, ſequitur functionem y fieri minimum ſecundae ſpeciei poſito $t = 0$ ſeu $x = f$. Quan-

quam ergo his caſibus termini $dt\sqrt[4]{dt}$ & $dt^2\sqrt[8]{dt}$ prae primo $\sqrt{dt}$ evaneſcant; tamen eorum ratio eſt habenda, ſi multiplicitas valorum ſpectetur, ut imaginaria evitentur.

EXEMPLUM VI.

Invenire differentiale functionis:

$$y = ax + bxx + (x - f)^n + (x - f)^{m + \frac{1}{2}n}$$

caſu $x = f$.

Si ponatur $x = f$ fiet $y = af + bff$, & ſi loco x ponatur $x + dx$ ſeu $f + dx$, prodibit valor proximus $y' = af + bff + adx + 2bfdx + bdx^2 + dx^n + dx^{m + \frac{1}{2}n}$, ita ut ſit $dy = adx + 2bfdx + bdx^2 + dx^n + dx^m\sqrt{dx^n}$. Niſi ergo ſit n numerus par, differentiale dx negative ſumi nequit. Ultimus autem terminus $dx^m\sqrt{dx^n}$ ſignum habet ambiguum; unde valor ipſius y' erit duplex uterque maior quam ipſius y, ſi quidem $a + 2bf$ fuerit quantitas affirmativa, atque exponentes n & $m + \frac{1}{2}n$ unitate fuerint maiores. Fiet ergo valor functionis y caſu $x = f$ minimus: hocque evenit ſive n ſit numerus integer ſive fractus, dummodo numerator hoc caſu, & ipſe numerus illo caſu non fuerit par.

351. Imprimis autem haec methodus differentialia ex ipſis principiis deducendi uſum habet in functionibus tranſcendentibus, cum quibuſdam caſibus differentiale more conſueto inventum vel evaneſcit, vel in infinitum excreſcere videtur. Occurrunt autem hic eiuſmodi infinitorum & infinite parvorum ſpecies, quae in algebraicis nunquam inveniuntur. Cum enim ſi i denotet numerum infinitum, li ſit quoque infinitus quidem, ſed tamen ad ipſum numerum i, eiuſque adeo poteſtatem quamcunque i^n, quamtumvis exiguus ſtatuatur exponens n rationem tenens infinite parvam, erit fractio $\frac{li}{i^n}$ infinite parva, neque ante finita eſſe poterit, quam exponens n fiat infinite parvus. Erit ergo li homogeneum cum i^n, ſi exponens n fuerit infinite parvus. Ponamus nunc $i = \frac{1}{\omega}$,

exi-

exiſtente ω quantitate infinite parva, erit $-l\omega$ homogeneum cum $\frac{1}{\omega^n}$, ſi exponens n ſit infinite parvus, ideoque $-\frac{1}{l\omega}$ homogeneum erit cum ω^n; hincque $-\frac{1}{ldx}$, erit infinite parvum comparandum cum dx^n, exiſtente n fractione infinite parva. Ita ſi fuerit $y = -\frac{1}{lx}$ differentiale ipſius y caſu $x = 0$, erit $= -\frac{1}{ldx} = dx^n$ ideoque dy ad dx atque ad quamcunque ipſius dx poteſtatem tenebit rationem infinitam: atque prae $-\frac{1}{ldx}$ evaneſcunt omnes omnino poteſtates ipſius dx, quantumvis exigui fuerint earum exponentes.

352. Deinde quoque vidimus, ſi a fuerit numerus unitate maior, & i infinitus, tum a^i fore infinitum tam excelſi gradus, ut prae eo non ſolum i, ſed etiam quaevis ipſius i poteſtas evaneſcat; neque i^n ante homogeneum cum a^i evadet, quam exponens n in infinitum fuerit auctus. Sit nunc $i = \frac{1}{\omega}$, ita ut ω infinite parvum denotet, erit $a^{\frac{1}{\omega}}$ homogeneum cum $\frac{1}{\omega^n}$, exiſtente n numero infinite magno: ideoque $a^{\frac{-1}{\omega}}$ ſeu $\frac{1}{a^{1:\omega}}$, erit infinite parvum comparandum cum ω^n. Hinc $\frac{1}{a^{1:dx}}$, erit infinite parvum, quod autem prae omnibus ipſius dx poteſtatibus evaneſcit; cum homogeneum ſit cum poteſtate dx^n exiſtente n numero infinite magno. Quare ſi quaeratur differentiale ipſius $y = \frac{1}{a^{1:x}}$ caſu $x = 0$; quoniam ſit $y = 0$, erit $dy = \frac{1}{a^{1:dx}}$, ideoque infinities minus eſt quam poteſtas quantumvis alta ipſius dx.

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353. Sin autem a ſit numerus unitate minor, tum quia $\frac{1}{a}$ ſit unitate maior, quaeſtio ad caſum praecedentem reducitur. Scilicet ſi habeatur expreſſio $a^{\frac{1}{\omega}}$, ea ponendo $a = 1 : b$ transſmutabitur in $b^{-\frac{1}{\omega}}$, ſeu $\frac{1}{b^{1:\omega}}$, quae homogenea erit ob $b > 1$ cum ω^n, exiſtente n numero infinite magno. His igitur praemiſſis ſequentia exempla reſolvere poterimus.

EXEMPLUM I.

Invenire differentiale functionis: $y = xx - \frac{1}{lx}$, *caſu* $x = 0$.

Quoniam poſito $x = 0$ fit $y = 0$, ſi ponamus $x + dx$, ſeu $0 + dx$ loco x, fiet $y' = dy = dx^2 - \frac{1}{ldx}$. Cum autem $-\frac{1}{ldx}$ homogeneum ſit cum dx^n, denotante n numerum infinite parvum, prae eo dx^2 evaneſcet, eritque $dy = -\frac{1}{ldx} = dx^n$. At vero quia logarithmi numerorum negativorum ſunt imaginarii, dx negative accipi non poterit; eritque adeo caſu $x = 0$ functio y mininum, ſed neque ad primam neque ad ſecundam ſpeciem pertinens. Ad primam ſcilicet ſpeciem non pertinet, quia y nullos habet valores antecedentes proximos, ſed tantum minus eſt valoribus ſequentibus, ſi x nihilo maius ſtatuatur. Ad ſecundam autem ſpeciem ideo non pertinet, quia valores ſequentes, quibuſcum comparatur, non ſunt gemini: ſic itaque prodit tertia ſpecies maximorum minimorumve, quae in functionibus logarithmicis & tranſcendentibus tantum locum habet, in algebraicis autem nunquam occurrit; de qua in ſequente parte de lineis curvis fuſius agetur.

EXEM-

EXEMPLUM II.

Invenire differentiale functionis: $y = (a - x)^n - x^n (la - lx)^n$ *caſu quo* $x = a$.

Differentiale hoc ſi n ſit numerus integer, ex formula generali $dy = pdx + \frac{1}{2} qdx^2 + \frac{1}{6} rdx^3 +$ &c. inveniri poteſt, erit enim:

$$pdx = -n(a-x)^{n-1} dx - nx^{n-1} dx (la - lx)^n + nx^{n-1} (la - lx)^{n-1} dx$$

qui valor poſito $x = a$ utique evaneſcit: nam etiamſi ſit $n = 1$, erit $pdx = - dx + dx = 0$. Si igitur ulterius progrediamur, erit: $\frac{1}{2} qdx^2 =$

$$\frac{n(n-1)}{1 \cdot 2}(a-x)^{n-2}dx^2 - \frac{n(n-1)}{1 \cdot 2} x^{n-2} dx^2 (la-lx)^n + \frac{n^2}{2} x^{n-2} dx^2 (la-lx)^{n-1}$$
$$+ \frac{n(n-1)}{1 \cdot 2} x^{n-2} dx^2 (la-lx)^{n-1} - \frac{n(n-1)}{1 \cdot 2} x^{n-2} dx^2 (la-lx)^{n-2}$$

Hinc ergo ſi fuerit $n = 1$, erit $\frac{1}{2} qdx^2 = \frac{dx^2}{2a}$ poſito $x = a$.

Simili modo ſi ſit $n = 2$, ad terminum tertium $\frac{1}{6} rdx^3$ eſſet pergendum, & ita porro. Facilius ergo utemur ipſis differentiationis principiis, & cum poſito $x = a$ fiat $y = 0$, ſi ponamus $x + dx$ ſeu $a + dx$ loco x, erit

$$y' = (-dx)^n - (a + dx)^n [la - l(a + dx)]^n = y + dy = dy \text{ ob } y = 0,$$

Eſt vero $l(a + dx) = la + \frac{dx}{a} - \frac{dx^2}{2a^2} + \frac{dx^3}{3a^3} -$ &c. unde fit

$$dy = (-dx)^n - \left(a^n + na^{n-1} dx + \frac{n(n-1)}{1 \cdot 2} a^{n-2} dx^2\right) \times$$
$$\left(-\frac{dx}{a} + \frac{dx^2}{2a^2} - \frac{dx^3}{3a^3}\right)^n = \frac{n}{2a}(-dx)^{n+1}$$

Caſu igitur $x = a$ erit formulae propoſitae differentiale quaeſitum dy, ut ſequitur:

ſi

ſi $n = 1$	$dy = \frac{dx^2}{2a}$ ut ante invenimus
ſi $n = 2$	$dy = -\frac{2dx^3}{2a}$
ſi $n = 3$	$dy = \frac{3dx^4}{2a}$
ſi $n = 4$	$dy = -\frac{4dx^5}{2a}$
&c.	&c.

Si ergo n fuerit numerus impar, functio y caſu $x = a$ fit minimum, ſin autem n ſit numerus par, neque maximum neque minimum: quod idem valet, ſi n fuerit fractio denominatorem habens imparem. Sin autem n fuerit fractio denominatorem habens parem, tum dx negative accipi debet, ne in imaginaria incidamus; & ob ambiguitatem ſignificationis functio quoque neque maxima neque minima evadet.

EXEMPLUM III.

Invenire differentiale functionis: $y = x^x$ *caſu* $x = \frac{1}{e}$ *denotante* e *numerum, cuius logarithmus hyperbolicus eſt* $= 1$.

Quia fit in genere $dy = x^x dx(lx + 1)$, hoc differentiale caſu $x = \frac{1}{e}$ ſeu $l\,x = -1$ evaneſcit. Comparetur ergo hoc differentiale cum forma generali $pdx + \frac{1}{2}qdx^2 +$ &c. erit $p = x^x(lx + 1)$ & $q = x^x(lx + 1)^2 + x^{x-1}$, & poſito $lx = -1$ ſeu $x = \frac{1}{e}$, erit $q = \left(\frac{1}{e}\right)^{\frac{1-e}{e}} = e^{\frac{e-1}{e}}$. Quare differentiale quaeſitum erit $dy = \frac{1}{2}e^{(e-1):e}dx^2$, evaditque ergo functio $y = x^x$ minimum caſu $x = \frac{1}{e}$.

EXEM-

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EXEMPLUM IV.

Invenire differentiale functionis huius: $y = x^n + e^{-1:x}$ *caſu quo* $x = o$.

Quia facto $x = o$ fit $y = o$, ſi ponatur $x = o + dx$, erit $y' = dy = dx^n + \frac{1}{e^{1:dx}}$. Vidimus autem $\frac{1}{e^{1:dx}}$ homogeneum eſſe cum poteſtate ipſius dx infinita, ſeu cum dx^{∞}, ideoque prae dx^n evaneſcet; ita ut ſit $dy = dx^n$.

354. Quod in differentialibus primis certis caſibus uſu venit, ut conſueta differentiationis regula non prodeant, idem quoque in differentialibus ſecundi ac tertii ſuperiorumque ordinum evenit, iis caſibus, quibus in forma differentiali completa:

$$d.y = pdx + \tfrac{1}{2} qdx^2 + \tfrac{1}{6} rdx^3 + \tfrac{1}{24} sdx^4 + \&c.$$

quantitatum q, r, s, &c. nonuullae vel evaneſcunt, vel in infinitum abeunt. Scilicet cum ſit:

$$dd.y = qdx^2 + rdx^3 + \tfrac{7}{12} sdx^4 + \&c.$$

ſi quo caſu fiat $q = o$, tum erit $ddy = rdx^3$; ſin autem eodem caſu & r evaneſcat, tum erit $ddy = \frac{7}{12} sdx^4$, & ita porro. Sin autem vel q vel r vel s &c. fiat infinitum, tum ex iſta ſerie differentiale ſecundum prorſus inveniri nequit, ſed confugiendum erit ad principia differentialium: ſcilicet ponendo $x + dx$ loco x quaeratur valor y', & ponendo $x + 2dx$ loco x valor ipſius y'', quo facto erit verus valor differentialis ſecundi $ddy = dy' - dy = y'' - 2y' + y$. Simili modo ſi de differentiali tertio quaeſtio proponatur, tum praeterea in y loco x ſcribatur $x + 3dx$, inventoque valore y''' erit $d^3y = y''' - 3y'' + 3y' - y$, ſicque deinceps. Quos caſus ſequentibus exemplis illuſtrabimus.

EXEM-

EXEMPLUM I.

Invenire differentiale ſecundum functionis $y = \frac{aa - xx}{aa + xx}$ *caſu quo ponitur* $x = \frac{a}{\sqrt{3}}$.

Quaerendo differentiale completum ipſius y, ex forma $dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 +$ &c. prodibunt pro p, q, r, s &c. ſequentes valores: $p = -\frac{4aax}{(aa+xx)^2}$; $q = -\frac{4a^4 + 12aaxx}{(aa+xx)^3}$; atque $r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}$.
Cum nunc ſit $ddy = qdx^2 + rdx^3 + \frac{7}{12}sdx^4 +$ &c. ob $q = 0$ caſu $x = \frac{a}{\sqrt{3}}$, eodemque caſu ſit $r = \frac{27\sqrt{3}}{8a^3}$, fiet differentiale ſecundum quaeſitum $ddy = \frac{27dx^3\sqrt{3}}{8a^3}$.

EXEMPLUM II.

Invenire differentiale tertium functionis $y = \frac{aa - xx}{aa + xx}$ *caſu* $x = a$.

Quaerendo ut ante differentiale completum $dy = \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 +$ &c. quia eſt differentiale tertium $d^3y = rdx^3 + \frac{3}{2}sdx^4$, ob $r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}$, fiet $r = 0$ caſu $x = a$; quare ad valorem s eſt progrediendum, qui erit: $s = \frac{48a^4 - 144aaxx}{(aa+xx)^4} - \frac{8x(48a^4x - 48aax^3)}{(aa+xx)^5}$ facto ergo $x = a$, erit $s = -\frac{96a^4}{24a^8} = -\frac{6}{a^4}$; unde hoc caſu erit $d^3y = -\frac{9dx^4}{a^4}$.

EXEM-

EXEMPLUM III.

Invenire differentialia cuiuſque gradus functionis
$y = ax^m + bx^n$ *caſu* $x = 0$.

Ponendo ſucceſſive $x + dx$; $x + 2dx$; $x + 3dx$; &c. loco x valores ſequentes functionis y erunt:

$$y' = a(x + dx)^m + b(x + dx)^n$$
$$y'' = a(x + 2dx)^m + b(x + 2dx)^n$$
$$y''' = a(x + 3dx)^m + b(x + 3dx)^n$$
&c.

Poſito ergo $x = 0$, erit $y = 0$, eiuſque differentialia erunt:

$$dy = adx^m + bdx^n$$
$$ddy = (2^m - 2)adx^m + (2^n - 2)bdx^n$$
$$d^3y = (3^m - 3.2^m + 3)adx^m + (3^n - 3.2^n + 3)bdx^n$$
$$d^4y = (4^m - 4.3^m + 6.2^m - 4)adx^m + (4^n - 4.3^n + 6.2^n - 4)bdx^n$$
&c.

Si igitur exponens n fuerit maior quam m, termini ſecundi in his expreſſionibus evaneſcunt prae primis. Interim tamen eorum ratio erit habenda, ſi n fuerit numerus fractus, ut caſus, quibus haec differentialia vel fiunt imaginaria, vel ambigua, diiudicari queant. Ulteriorem vero horum caſuum evolutionem in doctrinam de lineis curvis reſervari convenit.

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CAPUT XV.

DE VALORIBUS FUNCTIONUM, QUI CERTIS CASIBUS VIDENTUR INDETERMINATI.

355.

Si functio ipſius x quaecunque y fuerit fractio $\frac{P}{Q}$, cuius numerator ac denominator poſito loco x certo quodam valore ſimul evaneſcant; tum iſto caſu fractio $\frac{P}{Q}$ valorem functionis y exprimens evadet $= \frac{0}{0}$, quae expreſſio cum cuique quantitati ſive finitae ſive infinitae ſive infinite parvae poſſit eſſe aequalis, ex ea prorſus valor ipſius y hoc caſu colligi nequit, atque ideo videtur indeterminatus. Interim tamen facile perſpicitur, quia praeter hunc caſum functio y perpetuo valorem determinatum recipit, quicquid pro x ſubſtituatur, etiam hoc caſu valorem ipſius y indeterminatum eſſe non poſſe. Manifeſtum hoc fiet vel ex hoc exemplo, ſi fuerit $y = \frac{aa - xx}{a - x}$, quo facto $x = a$ fit utique $y = \frac{0}{0}$. Cum autem numeratore per denominatorem diviſo fiat $y = a + x$, evidens eſt ſi ponatur $x = a$ fore $y = 2a$, ita ut hoc caſu fractio illa $\frac{0}{0}$ aequivaleat quantitati $2a$.

356. Quoniam ergo ſupra oſtendimus, inter cyphras rationem quamcunque intercedere poſſe, in huiuſmodi exemplis ratio determinata, quam numerator ad denominatorem teneat, inveſtigari debet. Cum autem in cyphris abſolutis iſta diverſitas perſpici nequeat, earum loco quantitates infinite parvae intro-

introduci debent, quae etſi ratione ſignificationis a cyphra non differunt, tamen ex diverſis earum functionibus, quae numeratorem & denominatorem conſtituunt, valor fractionis ſponte elucet. Sic ſi habeatur iſta fractio $\frac{a\,dx}{b\,dx}$, etiamſi revera numerator & denominator ſit $=0$, tamen patet valorem huius fractionis eſſe determinatum nempe $=\frac{a}{b}$. Sin autem habeatur haec fractio $\frac{a\,dx^2}{b\,dx}$, huius valor erit nullus, quemadmodum huius valor $\frac{a\,dx}{b\,dx^2}$ eſt infinite magnus. Si igitur loco nihilorum, quae ſaepenumero in calculum ingrediuntur, infinite parva introducamus, hunc inde fructum percipiemus, ut rationem, quam illa nihila inter ſe tenent, mox cognoſcamus, nullumque amplius dubium circa ſignificationem huiuſmodi expreſſionum ſuperſit.

357. Quo haec planiora reddantur, ponamus fractionis $y=\frac{P}{Q}$ tam numeratorem quam denominatorem evaneſcere, ſi ſtatuatur $x=a$. Ad haec autem nihila, quae inter ſe comparari non poſſunt, evitanda, ponamus $x=a+dx$, quae poſitio revera in priorem $x=a$ recidit ob $dx=0$. Cum vero, ſi loco x ponatur $x+dx$, functiones P & Q abeant in $P+dP$ & $Q+dQ$; poſitioni $x=a+dx$ ſatisfiet, ſi in his valoribus ubique ſtatuatur $x=a$, quo quidem caſu P & Q evaneſcere aſſumuntur. Hinc ſi loco x ponatur $a+dx$, fractio $\frac{P}{Q}$ traſmutabitur in hanc $\frac{dP}{dQ}$, quae propterea valorem functionis $y=\frac{P}{Q}$ exprimit caſu $x=a$. Haecque expreſſio indeterminata amplius eſſe non poterit, ſi quidem functionum P & Q differentialia vera ſumantur, uti in capite praecedente do-

docuimus. Hoc enim pacto differentialia dP & dQ nunquam in nihilum abſolutum abeunt, ſed niſi per differentiale dx ipſum exprimantur, ſaltem per eius poteſtates exhibebuntur. Quodſi igitur reperiatur $dP = Rdx^m$ & $dQ = Sdx^n$, erit functionis $y = \frac{P}{Q}$ caſu $x = a$ valor $= \frac{Rdx^m}{Sdx^n}$, qui propterea erit finitus & $= \frac{R}{S}$, ſi fuerit $m = n$; ſin autem ſit $m > n$, tum valor fractionis propoſitae revera erit $= 0$: at ſi ſit $m < n$, iſte valor in infinitum excreſcit.

358. Quoties ergo huiuſmodi fractio occurrit $\frac{P}{Q}$, cuius numerator & denominator certo caſu puta $x = a$ ſimul evaneſcant, valor iſtius fractionis hoc caſu $x = a$ per ſequentem regulam invenietur:

Quaerantur quantitatum P *&* Q *differentialia caſu* $x = a$, *eaque loco ipſarum* P *&* Q *ſubſtituantur, quo facto fractio* $\frac{dP}{dQ}$ *exhibebit valorem fractionis* $\frac{P}{Q}$ *quaeſitum.*

Si differentialia dP & dQ methodo conſueta inventa neque infinita fiant neque evaneſcant caſu $x = a$, tum ea retineri poterunt; ſin autem ambo vel $= 0$ fiant vel $= \infty$, tum modo in praecedente Capite expoſito haec differentialia completa caſu $x = a$ inveſtigari debent. Plerumque etiam calculus mirifice contrahitur, ſi antea ponatur $x - a = t$ ſeu $x = a - t$, quo prodeat fractio $\frac{P}{Q}$, cuius numerator ac denominator evaneſcunt caſu $t = 0$; tum enim differentialia dP & dQ habebuntur, ſi ubique dt loco t ſubſtituatur.

EXEMPLUM I.

Quaeratur valor fractionis huius $\frac{b - \sqrt{(bb - tt)}}{tt}$ *caſu* $t = 0$.

Quoniam hoc caſu $t = 0$ & numerator & denominator evaneſcit,

neſcit, loco t tantum ſcribatur dt, atque valor quaeſitus exprimetur hac fractione $\frac{b - \surd(bb - dt^2)}{dt^2}$. Cum vero ſit $\surd(bb - dt^2) = b - \frac{dt^2}{2b}$, iſta fractio abit in hanc $\frac{dt^2}{2bdt^2} = \frac{1}{2b}$. Hinc fractio propoſita $\frac{b - \surd(bb - tt)}{tt}$ caſu $t = 0$ recipit hunc valorem $\frac{1}{2b}$.

EXEMPLUM II.

Quaeratur valor huius fractionis:

$$\frac{\surd(aa + ax + xx) - \surd(aa - ax + xx)}{\surd(a + x) - \surd(a - x)} \quad \textit{caſu} \ x = 0.$$

Hic iterum ſtatim dx loco x ſubſtitui poteſt; quo facto cum ſit:

$$\surd(aa + adx + dx^2) = a + \tfrac{1}{2}dx + \frac{3dx^2}{8a};$$

$$\surd(aa - adx + dx^2) = a - \tfrac{1}{2}dx + \frac{3dx^2}{8a};$$

atque $\surd(a + dx) = \surd a + \frac{dx}{2\surd a}$; $\surd(a - dx) = \surd a - \frac{dx}{2\surd a}$ fiet numerator $= dx$ & denominator $= \frac{dx}{\surd a}$, ex quo fractionis propoſitae valor quaeſitus erit $= \surd a$.

EXEMPLUM III.

Quaeratur valor huius fractionis:

$$\frac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\surd(2ax - aa)}{xx - 2ax - aa + 2a\surd(2ax - xx)} \quad \textit{caſu} \ x = a.$$

Si more conſueto differentialia ſumantur & in loca numeratoris ac denominatoris ſubſtituantur, habebitur:

$\frac{3xx - 8ax + 7a^2 - 2a^3 : \surd(2ax - aa)}{2x - 2a + 2a(a - x) : \surd(2ax - xx)}$, cuius fractionis numerator

merator ac denominator denuo evaneſcunt, ſi ponatur $x = a$. Quare ob eandem rationem eorum loco denuo ipſorum differentialia ſubſtituantur, prodibitque: $\frac{6x - 8a + 2a^4 : (2ax - aa)^{\frac{3}{2}}}{2 - 2a^3 : (2ax - xx)^{\frac{3}{2}}}$ cuius numerator ac denominator iterum caſu $x = a$ evaneſcunt. Pergamus ergo eorum loco ipſorum differentialia ſubſtituere: $\frac{6 - 6a^5 : (2ax - aa)^{\frac{5}{2}}}{6a^3(a-x) : (2ax - xx)^{\frac{5}{2}}} = \frac{1 - a^5 : (2ax - aa)^{\frac{5}{2}}}{a^3(a-x) : (2ax - xx)^{\frac{5}{2}}}$. Verum & hic poſito $x = a$ denuo tam numerator quam denominator evaneſcunt. Porro igitur differentialibus ipſorum loco ſubſtitutis, orietur:

$$\frac{5a^6 : (2ax - aa)^{\frac{7}{2}}}{-(5a^5 - 8a^4x + 4a^3xx) : (2ax - xx)^{\frac{7}{2}}}.$$

Nunc denique loco x ponatur a, prodibitque haec fractio determinata $\frac{5 : a}{-1 : a^2} = -5a$, qui eſt valor quaeſitus fractionis propoſitae:

Quodſi autem antequam haec inveſtigatio ſuſcipiatur, ponatur $x = a + t$, fractio propoſita transmutabitur in hanc: $\frac{2a^3 + 2a^2t - att + t^3 - 2a^2\surd(aa + 2at)}{-2aa + tt + 2a\surd(aa - tt)}$ quae cum recipiat formam $\frac{0}{0}$, ſi ponatur $t = 0$, ponatur dt loco t, & erit: $\frac{2a^3 + 3a^2dt - adt^2 + dt^3 - 2a^2\surd(aa + 2adt)}{-2aa + dt^2 + 2a\surd(aa - dt^2)}$. Convertantur iam formulae irrationales in ſeries, quae eouſque continuentur, quoad termini a membro rationali non amplius deſtruantur: $\surd(aa + 2adt) = a + dt - \frac{dt^2}{2a} + \frac{dt^3}{2aa} - \frac{5dt^4}{8a^3}$

$$\surd(aa - dt^2) = a - \frac{dt^2}{2a} - \frac{dt^4}{8a^3},$$

qui-

quibus valoribus ſubſtitutis prodibit fractio haec:

$$\frac{5dt^4:4a}{-dt^4:4aa} = -5a,$$

qui eſt valor fractionis propoſitae iam ante inventus.

EXEMPLUM IV.

Invenire valorem huius fractionis:

$$\frac{a+\surd(2aa-2ax)-\surd(2ax-xx)}{a-x+\surd(aa-xx)} \quad \textit{caſu } x = a.$$

Subſtitutis in loca numeratoris & denominatoris eorum differentialibus prodibit haec fractio, quae caſu $x = a$ ipſi propoſitae erit aequalis: $\frac{-a:\surd(2aa-2ax)-(a-x):\surd(2ax-xx)}{-1-x:\surd(aa-xx)}$ cuius numerator ac denominator caſu $x = a$ fiunt infiniti. Verum ſi uterque per $-\surd(a-x)$ multiplicetur, habebitur

$$\frac{a:\surd 2a+(a-x)^{\frac{3}{2}}:\surd(2ax-xx)}{\surd(a-x)+x:\surd(a+x)}$$ quae poſito $x = a$ dabit hunc valorem determinatum $\frac{a:\surd 2a}{a:\surd 2a} = 1$, qui propterea aequalis eſt fractioni propoſitae caſu $x = a$.

359. Si igitur habeatur fractio $\frac{P}{Q}$, cuius numerator & denominator caſu $x = a$ evaneſcat, eius valor per conſuetas differentiandi regulas aſſignari poterit, neque opus erit ad differentialia, quae capite praecedente tractavimus, recurrere. Sumtis enim differentialibus fractio propoſita $\frac{P}{Q}$ caſu $x = a$ aequalis erit fractioni $\frac{dP}{dQ}$; cuius ſi numerator & denominator poſito $x = a$ induant valores finitos, cognoſcetur valor fractionis propoſitae; ſin autem alter fiat $= 0$, manente altero finito, tum fractio erit vel $= 0$ vel $= \infty$, prout vel numerator

merator evaneſcat vel denominator. At ſi alteruter vel uterque fiat $=\infty$, quod evenit, ſi dividantur per quantitates caſu $x=a$ evaneſcentes; tum multiplicando utrumque per hos diviſores, iſtud incommodum tolletur, uti in exemplo poſtremo evenit. Quodſi vero tam numerator quam denominator caſu $x=a$ denuo evaneſcat, tum iterum, uti initio factum eſt, differentialia erunt capienda, ita ut haec fractio $\frac{ddP}{ddQ}$ prodeat, quae caſu $x=a$ propoſitae adhuc erit aequalis, &, ſi idem rurſus in hac fractione uſu veniat, ut fiat $=\frac{0}{0}$, tum in eius locum ſurrogetur haec $\frac{d^3 P}{d^3 Q}$, atque ita porro, donec ad fractionem perveniatur, quae valorem determinatum exhibeat, ſive finitum ſive infinite magnum ſive infinite parvum. Sic in exemplo tertio oportebat ad fractionem $\frac{d^4 P}{d^4 Q}$ progredi, antequam valorem fractionis propoſitae $\frac{P}{Q}$ aſſignari licuerit.

360. Uſus huius inveſtigationis elucet in definiendis ſummis ſerierum, quas ſupra Capite II. §. 22. eruimus, ſi ponatur $x=1$. Ex iis enim, quae ibi tradita ſunt, ſequitur fore:

$$x+x^2+x^3+\dots+x^n = \frac{x-x^{n+1}}{1-x}$$

$$x+x^3+x^5+\dots+x^{2n-1} = \frac{x-x^{2n+1}}{1-xx}$$

$$x+2x^2+3x^3+\dots+nx^n = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

$$x+3x^3+5x^5+\dots+(2n-1)x^{2n-1} = \frac{x+x^3-(2n+1)x^{2n+1}+(2n-1)x^{2n+3}}{(1-xx)^2}$$

$$x+4x^2+9x^3+\dots+n^2x^n = \frac{x+x^2-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}$$

&c.

Quod

Quod ſi nunc harum ſerierum ſummae deſiderentur caſu quo $x = 1$, in expreſſionibus iſtis tam numerator quam denominator evaneſcunt. Valores ergo harum ſummarum caſu $x = 1$ methodo hic expoſita definiri poterunt. Quoniam vero eaedem ſummae aliunde conſtant, ex conſenſu veritas huius methodi magis elucebit.

EXEMPLUM I.

Definire valorem huius fractionis $\frac{x - x^{n+1}}{1 - x}$ *caſu* $x = 1$, *qui exhibebit ſummam ſeriei* $1 + 1 + 1 + \ldots + 1$ *ex* n *terminis conſtantis, quae propterea erit* $= n$.

Quoniam caſu $x = 1$ numerator ac denominator evaneſcit, ſubſtituantur differentialia in eorum locum, habebiturque $\frac{1 - (n+1)x^n}{-1}$, quae poſito $x = 1$ dat n pro ſumma ſeriei quaeſita.

EXEMPLUM II.

Definire valorem fractionis $\frac{x - x^{2n+1}}{1 - xx}$ *caſu* $x = 1$, *qui exhibebit ſummam ſeriei* $1 + 1 + 1 + \ldots + 1$ *ex* n *terminis conſtantis, quae propterea erit* $= n$.

Sumtis differentialibus fractio propoſita tranſmutatur in hanc: $\frac{1 - (2n+1)x^{2n}}{-2x}$, cuius valor poſito $x = 1$, erit $= n$.

EXEMPLUM III.

Invenire valorem huius fractionis: $\frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$ *caſu* $x = 1$, *qui exprimet ſummam ſeriei* $1 + 2 + 3 + \ldots + n$, *quam conſtat eſſe* $= \frac{nn + n}{2}$.

Sumtis differentialibus pervenietur ad hanc fractionem

 1 —

$\frac{1-(n+1)^2 x^n + n(n+2)x^{n+1}}{-2(1-x)}$, cuius adhuc tam numerator quam denominator caſu $x=1$ evaneſcit. Hinc denuo differentialia ſumantur, ut prodeat haec fractio:

$$\frac{-n(n+1)^2 x^{n-1} + n(n+1)(n+2)x^n}{2}, \text{ quae poſito } x=1$$

abit in $\frac{n(n+1)}{2} = \frac{nn+n}{2}$ ſummam ſeriei propoſitae.

EXEMPLUM IV.

Invenire valorem huius fractionis:

$$\frac{x + x^3 - (2n+1)x^{2n+1} + (2n-1)x^{2n+3}}{(1-xx)^2} \quad \textit{caſu } x=1,$$

qui exprimet ſummam ſeriei $1+3+5+\dots+(2n-1)$ *quam conſtat eſſe* $= nn$.

Subſtitutis differentialibus in loca numeratoris & denominatoris provenit haec fractio:

$$\frac{1 + 3xx - (2n+1)^2 x^{2n} + (2n-1)(2n+3)x^{2n+2}}{-4x(1-xx)}$$

quae cum adhuc idem incommodum habeat, ut poſito $x=1$ abeat in $\frac{0}{0}$, denuo differentialia ſumantur,

$$\frac{6x - 2n(2n+1)^2 x^{2n-1} + (2n-1)(2n+2)(2n+3)x^{2n+1}}{-4+12xx}$$

quae poſito $x=1$ abit in:

$$\frac{6 - 2n(2n+1)^2 + (2n-1)(2n+2)(2n+3)}{8} = nn.$$

EXEMPLUM V.

Invenire valorem huius fractionis:

$$\frac{x + x^2 - (n+1)^2 x^{n+1} + (2nn + 2n - 1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$

caſu $x=1$, *qui dabit ſummam ſeriei* $1+4+9+\dots+n^2$, *quam conſtat eſſe* $= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

Sumtis numeratoris ac denominatoris differentialibus, fiet

$$\frac{1+2x-(n+1)^3x^n+(n+2)(2nn+2n-1)x^{n+1}-nn(n+3)x^{n+2}}{-3(1-x)^2}$$

in qua cum numerator ac denominator poſito $x = 1$ denuo evaneſcat, differentialia ſecunda ſumantur:

$$\frac{2-n(n+1)^3x^{n-1}+(n+1)(n+2)(2nn+3n-1)x^n-n^2(n+2)(n+3)x^{n+1}}{6(1-x)}$$

eodem vero adhuc ſubſiſtente incommodo, ad differentialia tertia procedatur, ut prodeat haec fractio:

$$\frac{-n(n-1)(n+1)^3x^{n-2}+n(n+1)(n+2)(2nn+2n-1)x^{n-1}-n^2(n+1)(n+2)(n+3)x^n}{-6}$$

quae tandem poſito $x = 1$ abit in hanc formam determinatam:

$$\frac{-n(n-1)(n+1)^3+n(n+1)(n+2)(nn-n-1)}{-6} = \frac{n(n+1)(2n+1)}{6} =$$

$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$; qui eſt ille ipſe valor, quo ſeriem memoratam exprimi invenimus.

EXEMPLUM VI.

Sit propoſita iſta fractio $\frac{x^m - x^{m+n}}{1 - x^{2p}}$, *cuius valorem caſu* $x = 1$ *aſſignari oporteat.*

Quoniam haec fractio eſt productum ex his duabus: $\frac{x^m}{1+x^p} \cdot \frac{1-x^n}{1-x^p}$, prioris autem factoris caſu $x = 1$ valor eſt $= \frac{1}{2}$, tantum opus eſt ut alterius factoris $\frac{1-x^n}{1-x^p}$ valor eodem caſu quaeratur, qui ſumtis differentialibus erit $= \frac{nx^{n-1}}{px^{p-1}} = \frac{n}{p}$: unde fractionis propoſitae valor caſu $x = 1$, erit $= \frac{n}{2p}$. Idem valor prodit, ſi immediate differentialia in fractione propoſita capiantur: fiet enim $\frac{mx^{m-1} - (m+n)x^{m+n-1}}{-2px^{2p-1}}$, cuius va-

 lor

lor poſito $x = 1$, erit $= \frac{-n}{-2p} = \frac{n}{2p}$, ut ante:

361. Eadem methodo erit utendum, ſi in fractione propoſita $\frac{P}{Q}$ vel numerator vel denominator vel uterque fuerit quantitas tranſcendens. Quae operationes, quo clarius explicentur, ſequentia exempla adiicere viſum eſt.

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{a^n - x^n}{la - lx}$, *cuius valor quaeratur caſu* $x = a$.

Sumtis differentialibus ſtatim pervenitur ad hanc fractionem $- \frac{nx^{n-1}}{-1:x} = nx^n$, cuius valor poſito $x = a$, erit na^n.

EXEMPLUM II.

Sit propoſita iſta fractio $\frac{lx}{\sqrt{(1-x)}}$, *cuius valor quaeritur caſu* $x = 1$.

Sumtis differentialibus numeratoris & denominatoris prodit $\frac{1:x}{-1:2\sqrt{(1-x)}} = \frac{-2\sqrt{(1-x)}}{x}$, cuius valor poſito $x = 1$, cum ſit $= 0$, ſequitur fractionem $\frac{lx}{\sqrt{(1-x)}}$ caſu $x = 1$ evaneſcere.

EXEMPLUM III.

Sit propoſita iſta fractio $\frac{a - x - ala + alx}{a - \sqrt{(2ax - xx)}}$, *cuius valor quaeratur poſito* $x = a$, *quo caſu numerator & denominator evaneſcunt*.

Differentiatis ſecundum regulam numeratore ac denominatore erit $\frac{-1 + a:x}{-(a-x):\sqrt{(2ax-xx)}} = \frac{(a-x)\sqrt{(2ax-xx)}}{-x(a-x)}$: ubi etſi numera-

merator ac denominator caſu $x = a$ adhuc evaneſcit, tamen quia uterque diviſibilis eſt per $a - x$, habebitur iſta fractio $-\sqrt{\frac{2a - x}{x}}$, cuius valor caſu $x = a$ eſt determinatus atque $= -1$; abitque igitur fractio propoſita in -1, ſi ponatur $x = a$.

EXEMPLUM IV.

Sit propoſita iſta fractio $\frac{e^x - e^{-x}}{l(1+x)}$, *cuius valor quaeritur poſito* $x = 0$.

Sumtis differentialibus habebitur iſta functio $\frac{e^x + e^{-x}}{1:(1+x)}$, quae poſito $x = 0$ dat 2 pro valore quaeſito.

EXEMPLUM V.

Invenire valorem huius fractionis $\frac{e^x - 1 - l(1+x)}{xx}$, *caſu quo ponitur* $x = 0$.

Si loco numeratoris ac denominatoris eorum differentialia ſubſtituantur, orietur haec fractio $\frac{e^x - 1:(1+x)}{2x}$, quae cum adhuc abeat in $\frac{0}{0}$, ſi ponatur $x = 0$, denuo differentialia ſumantur, ut habeatur $\frac{e^x + 1:(1+x)^2}{2}$, quae poſito $x = 0$ praebet $\frac{1+1}{2} = 1$. Quod idem patet ſi loco x ſtatim $0 + dx$ ſubſtituatur: cum enim ſit $e^{dx} = 1 + dx + \frac{1}{2}dx^2 + \&c.$ & $l(1+dx) = dx - \frac{1}{2}dx^2 + \&c.$, $\frac{e^{dx} - 1 - l(1+dx)}{dx^2} = \frac{dx^2}{dx^2} = 1$.

EXEM-

EXEMPLUM VI.

Quaeratur valor fractionis $\frac{x^n}{lx}$, *caſu quo ponitur* $x = \infty$.

Quo iſta fractio ad formam, quae hoc caſu tranſeat in $\frac{0}{0}$ reducatur, ita repraeſentetur $\frac{1:lx}{1:x^n}$, ſic enim caſu $x = \infty$, tam numerator quam denominator evaneſcet. Ponatur vero porro $x = \frac{1}{y}$, ita ut caſu $x = \infty$, fiat $y = 0$, atque proponetur iſta fractio $-\frac{1:ly}{y^n}$, cuius valor caſu $y = 0$ inveſtigari debet. Sumtis autem differentialibus erit $\frac{1:y(ly)^2}{ny^{n-1}} = \frac{1:(ly)^2}{ny^n}$, quae poſito $y = 0$, cum abeat in $\frac{0}{0}$, ſumantur denuo differentialia, eritque $\frac{-2:(ly)^3}{n^2y^n}$; ubi quia idem incommodum adeſt, ſi porro differentialia ſumantur prodibit $\frac{6:(ly)^4}{n^3y^n}$, ſicque quouſque procedamus, perpetuo idem incommodum occurret. Quamobrem ut hoc non obſtante valorem quaeſitum eruamus, ſit s valor fractionis $-\frac{1:ly}{y^n}$ caſu, quo ponitur $y = 0$, & cum eodem caſu ſit quoque $s = \frac{1:(ly)^2}{ny^n}$; erit ex illa aequatione $ss = \frac{1:(ly)^2}{y^{2n}}$, quae per iſtam diviſa dabit $s = \frac{ny^n}{y^{2n}}$ $= \frac{n}{y^n}$, ex qua perſpicitur caſu $y = 0$ fieri s infinitum. Fit ergo fractionis $-\frac{1:ly}{y^n}$ valor caſu $y = 0$ infinitus, ideoque poſito $y = dx$, habebit $\frac{1}{ldx}$ ad dx^n rationem infinitam, uti iam ſupra innuimus.

EXEM-

EXEMPLUM VII.

Quaeratur valor fractionis $\frac{x^n}{e^{-1:x}}$ *caſu* $x = 0$, *quo tam numerator quam denominator evaneſcit.*

Sit hoc caſu $\frac{x^n}{e^{-1:x}} = s$, erit ſumtis differentialibus quoque $s = \frac{nx^{n-1}}{e^{-1:x}:xx} = \frac{nx^{n+1}}{e^{-1:x}}$, & quia hic idem incommodum occurrit, perpetuoque recurrit, quouſque differentiationes continuentur, remedio ante adhibito utamur. Prior aequatio dat $x^n = e^{-1:x}s$ & $x^{n(n+1)} = e^{-(n+1:x)}s^{n+1}$ altera aequatio dat $x^{n+1} = e^{-1:x}s:n$, unde fit $x^{n(n+1)} = e^{-n:x}s^n:n^n$, qui valor illi aequatus dabit $e^{-1:x}sn^n = 1$ ideoque $s = \frac{1}{n^n e^{-1:x}} = \infty$, ſi $x = 0$. Quare poſito x infinite parvo habebit dx^n ad $e^{-1:dx}$ rationem infinite magnam, quicunque numerus finitus pro n ſtatuatur: unde ſequitur $e^{-1:dx}$ eſſe infinite parvum homogeneum cum dx^m, ſi m fuerit numerus infinite magnus.

EXEMPLUM VIII.

Quaeratur valor fractionis $\frac{1 - \sin x + \cos x}{\sin x + \cos x - 1}$ *caſu, quo ponitur* $x = \frac{\pi}{2}$ *ſeu arcui 90 graduum.*

Sumtis differentialibus obtinebitur haec fractio $\frac{-\cos x - \sin x}{\cos x - \sin x}$ quae poſito $x = \frac{\pi}{2}$ ob $\sin x = 1$ & $\cos x = 0$ abit in 1; ita ut unitas ſit valor quaeſitus fractionis propoſitae. Quod idem patet ſine differentiatione: cum enim ſit

$$\cos x = \sqrt{(1 + \sin x)(1 - \sin x)}$$

fractio

fractio propoſita abit in hanc $\frac{\sqrt{(1-\sin x)}+\sqrt{(1+\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}$, quae fit evidenter $=1$, ſi fiat ſin $x=1$.

EXEMPLUM IX.

Invenire valorem huius expreſſionis $\frac{x^x-x}{1-x+lx}$ *caſu quo ponitur* $x=1$.

Loco numeratoris & denominatoris eorum differentialibus ſubſtitutis prodibit iſta fractio : $\frac{x^x(1+lx)-1}{-1+1:x}$, quae cum etiam nunc fiat $=\frac{0}{0}$ poſito $x=1$, ſumantur denuo differentialia, ut prodeat $\frac{x^x(1+lx)^2+x^x:x}{-1:xx}$, quae poſito $x=1$ abit in -2, qui eſt valor fractionis propoſitae caſu $x=1$.

362. Quoniam hic omnes expreſſiones, quae quibuſdam caſibus indeterminatos valores recipere videntur, pertractare conſtituimus, huc non ſolum pertinent eae fractiones $\frac{P}{Q}$, quarum numerator ac denominator certo caſu evaneſcunt; ſed etiam eiuſmodi fractiones, quarum numerator ac denominator certo caſu fiunt infiniti, huc ſunt referendae: propterea quod earum valores aeque indeterminati videntur. Si ſcilicet P & Q eiuſmodi fuerint functiones ipſius x, ut caſu quopiam $x=a$, ambae fiant infinitae, fractioque $\frac{P}{Q}$ induat hanc formam $\frac{\infty}{\infty}$; quoniam infinita aeque ac cyphrae inter ſe rationem quamcunque tenere poſſunt, hinc valor verus minime cognoſci poteſt. Hic quidem caſus ad praecedentem revocari poteſt, fractionem $\frac{P}{Q}$ in hanc formam $\frac{1:Q}{1:P}$ transmutando, cuius fra-

fractionis nunc numerator ac denominator caſu $x = a$ evaneſcunt; ideoque eius valor modo ante tradito inveniri poteſt. At vero quoque ſine hac transformatione valor invenietur, ſi loco x non a, ſed $a + dx$ ſubſtituatur, quo facto non eiuſmodi infinita abſoluta ∞ provenient; ſed ita erunt expreſſa $\frac{1}{dx}$ vel $\frac{A}{dx^n}$; quae expreſſiones etſi ſunt aeque infinitae ac ∞, tamen comparatione inter dx eiuſve poteſtates inſtituta, valor quaeſitus facile colligetur.

363. Ad eandem claſſem quoque pertinent producta ex duobus factoribus conſtantia, quorum alter certo caſu $x = a$ evaneſcit, alter vero in infinitum abit: cum enim quaevis quantitas per huiuſmodi productum $0 . \infty$ repraeſentari poſſit, eius valor indefinitus videtur. Sit PQ huiuſmodi productum, in quo, ſi ponatur $x = a$, fiat $P = 0$ & $Q = \infty$, eius valor per praecepta ante tradita invenietur, ſi ponatur $Q = \frac{1}{R}$, tum enim productum PQ tranſmutabitur in fractionem $\frac{P}{R}$: cuius numerator ac denominator ante caſu $x = a$ evaneſcunt, ideoque eius valor methodo ante expoſita inveſtigari poterit. Sic ſi quaeratur valor huius producti $(1-x)$ tang $\frac{\pi x}{2}$ caſu $x = 1$, quo fit $1 - x = 0$ & tang $\frac{\pi x}{2} = \infty$, convertatur id in hanc fractionem $\frac{1-x}{\cot \frac{1}{2}\pi x}$, cuius numerator ac denominator caſu $x = 1$ evaneſcunt. Cum igitur ſit differentiale numeratoris $1 - x = -dx$, & differentiale denominatoris $\cot \frac{\pi x}{2} = -\frac{\pi dx : 2}{(\sin \frac{1}{2}\pi x)^2}$, caſu $x = 1$ valor fractionis propoſitae erit $= \frac{2}{\pi} . \sin \frac{\pi x}{2} . \sin \frac{\pi x}{2} = \frac{2}{\pi}$, ob $\sin \frac{\pi}{2} = 1$.

364. Imprimis autem huc ſunt referendae eiuſmodi expreſſiones, quae dum ipſi x certus quidam valor tribuitur, abeunt in huiuſmodi formam $\infty - \infty$: quoniam enim duo infinita quavis quantitate finita inter ſe diſcrepare poſſunt, manifeſtum eſt hoc caſu valorem expreſſionis non determinari, niſi differentia inter illa duo infinita aſſignari poſſit. Iſte ergo caſus occurrit, ſi proponatur huiuſmodi functio $P - Q$, in qua poſito $x = a$ fiat tam $P = \infty$ quam $Q = \infty$, quo caſu ope regulae ante traditae valor quaeſitus non tam facile aſſignari poteſt. Etſi enim poſito hoc caſu fieri $P - Q = f$, ſtatuatur $e^{P-Q} = e^f$, ita ut ſit $e^f = \frac{e^{-Q}}{e^{-P}}$, ubi caſu $x = a$ tam numerator e^{-Q} quam denominator e^{-P} evaneſcit; tamen ſi regula ante tradita huc transferatur, fiet $e^f = \frac{e^{-Q}dQ}{e^{-P}dP}$, unde ob $e^f = \frac{e^{-Q}}{e^{-P}}$, fieret $1 = \frac{dQ}{dP}$, ideoque valor quaeſitus ipſius f hinc non innoteſcit. Quoties quidem P & Q ſunt quantitates algebraicae, quoniam hae infinitae fieri nequeunt, niſi ſint fractiones, quarum denominatores evaneſcunt; tum $P - Q$ in unicam fractionem colligi poterit, cuius denominator pariter evaneſcet. Quo facto ſi etiam numerator evaneſcat, valor modo ſupra explicato definietur: ſin autem numerator non evaneſcat, tum eius valor revera erit infinitus. Sic ſi huius expreſſionis $\frac{1}{1-x} - \frac{2}{1-xx}$ valor deſideretur caſu $x = 1$, quia ea abit in $\frac{x-1}{1-xx} = \frac{-1}{1+x}$, patet valorem quaeſitum eſſe $= -\frac{1}{2}$.

365. Verum ſi functiones P & Q fuerint tranſcendentes, tum plerumque haec transformatio ad calculum moleſtiſſimum perduceret. Expediet ergo his caſibus methodo directa

ſta uti, atque loco $x = a$, quo ambae quantitates P & Q in infinitum abeunt, poni $x = a + \omega$, exiſtente ω quantitate infinite parva, pro qua dx accipi poterit. Quo facto ſi fiat $P = \frac{A}{\omega} + B$ & $Q = \frac{A}{\omega} + C$ manifeſtum eſt functionem $P - Q$ abituram eſſe in $B - C$, qui erit valor finitus. Rationem igitur huiuſmodi functionum valores inveſtigandi ſequentibus exemplis illuſtrabimus.

EXEMPLUM I.

Quaeratur valor huius expreſſionis $\frac{x}{x-1} - \frac{1}{lx}$ *caſu, quo ponitur* $x = 1$.

Quoniam tam $\frac{x}{x-1}$ quam $\frac{1}{lx}$ fit infinitum poſito $x = 1$, ſtatuatur $x = 1 + \omega$, atque expreſſio propoſita transformabitur in $\frac{1+\omega}{\omega} - \frac{1}{l(1+\omega)}$. Cum igitur ſit $l(1+\omega) = \omega - \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3 - \&c. = \omega(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \&c.)$ habebitur

$$\frac{(1+\omega)(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \&c.) - 1}{\omega(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \&c.)} = \frac{\frac{1}{2}\omega - \frac{1}{6}\omega^2 + \&c.}{\omega(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \&c.)}$$

$$= \frac{\frac{1}{2} - \frac{1}{6}\omega + \&c.}{1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \&c.}$$

Poſito nunc ω infinite parvo ſeu $\omega = 0$, manifeſtum eſt valorem quaeſitum eſſe $= \frac{1}{2}$.

EXEMPLUM II.

Denotantibus e *numerum cuius logarithmus hyperbolicus eſt* $= 1$, *&* π *ſemicircumferentiam circuli, cuius radius eſt* $= 1$, *inveſtigare valorem huius expreſſionis:*

$$\frac{\pi x - 1}{2xx} + \frac{\pi}{x(e^{2\pi x} - 1)}, \text{ caſu } x = 0.$$

Expreſſio iſta propoſita exhibet ſummam huius ſeriei:

$\frac{1}{1+xx}+\frac{1}{4+xx}+\frac{1}{9+xx}+\frac{1}{16+xx}+\frac{1}{25+xx}+$ &c. unde ſi ponatur $x=0$, prodire debet ſumma ſeriei huius $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+$ &c. quam conſtat eſſe $=\frac{\pi\pi}{6}$. Facto autem $x=0$ expreſſionis propoſitae $\frac{\pi x-1}{2xx}+\frac{\pi}{x(e^{2\pi x}-1)}$ valor maxime videtur indeterminatus, ob omnes terminos infinitos. Ponatur ergo $x=\omega$, exiſtente ω quantitate infinite parva, atque membrum prius $\frac{\pi x-1}{2xx}$ abit in $-\frac{1}{2\omega^2}+\frac{\pi}{2\omega}$. Cum deinde ſit $e^{2\pi\omega}-1=2\pi\omega+2\pi^2\omega^2+\frac{4}{3}\pi^3\omega^3+$ &c. alterum membrum $\frac{\pi}{x(e^{2\pi x}-1)}$ abit in

$$\frac{\pi}{\omega(2\pi\omega+2\pi^2\omega^2+\frac{4}{3}\pi^3\omega^3+\&c.)}=\frac{1}{2\pi^2(1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\&c.)}$$

At eſt $\frac{1}{1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\&c.}=1-\pi\omega+\frac{1}{3}\pi^2\omega^2-$ &c.

unde poſterius membrum fit $=\frac{1}{2\omega^2}-\frac{\pi}{2\omega}+\frac{1}{6}\pi^2-$ &c. ad quod ſi prius addatur prodit $\frac{1}{6}\pi^2$, qui eſt valor quaeſitus expreſſionis propoſitae caſu $x=0$.

Idem quoque per methodum fractionum, quarum numerator ac denominator certo caſu evaneſcunt, praeſtari poteſt: expreſſio enim propoſita in hanc fractionem tranſmutatur:

$$\frac{\pi xe^{2\pi x}-e^{2\pi x}+\pi x+1}{2xxe^{2\pi x}-2xx},$$

cuius numerator ac denominator caſu $x=0$ evaneſcunt. Sumtis ergo differentialibus oritur:

$$\frac{\pi e^{2\pi x}+2\pi\pi xe^{2\pi x}-2\pi e^{2\pi x}+\pi}{4xe^{2\pi x}+4\pi xxe^{2\pi x}-4x}$$

ſive

ſive haec $\dfrac{\pi - \pi e^{2\pi x} + 2\pi\pi x e^{2\pi x}}{4xe^{2\pi x} + 4\pi xx e^{2\pi x} - 4x}$

cuius, ſi ponatur $x = 0$, adhuc mmerator ac denominator evaneſcunt. Quare ſumtis denuo differentialibus habebitur:

$$\frac{-2\pi\pi e^{2\pi x} + 2\pi\pi e^{2\pi x} + 4\pi^3 x e^{2\pi x}}{4e^{2\pi x} + 8\pi x e^{2\pi x} + 8\pi x e^{2\pi x} + 8\pi^2 xx e^{2\pi x} - 4}$$

ſeu $$\frac{\pi^3 x e^{2\pi x}}{e^{2\pi x} + 4\pi x e^{2\pi x} + 2\pi^2 x^2 e^{2\pi x} - 1}$$

ſeu $$\frac{\pi^3 x}{1 + 4\pi x + 2\pi^2 x^2 - e^{-2\pi x}}$$

cuius numerator ac denominator adhuc evaneſcunt caſu $x = 0$. Quocirca iterum differentialia ſumantur

$$\frac{\pi^3}{4\pi + 4\pi^2 x + 2\pi e^{-2\pi x}}$$

quae fractio poſito $x = 0$ abit in $\frac{\pi^2}{6}$, ut ante.

EXEMPLUM III.

Retinentibus e *&* π *eoſdem valores, quaeratur valor expreſſionis huius* $\dfrac{\pi}{4x} - \dfrac{\pi}{2x(e^{\pi x} + 1)}$ *caſu* $x = 0$.

Expreſſio haec tranſmutatur in hanc: $\dfrac{\pi e^{\pi x} - \pi}{4xe^{\pi x} + 4x}$ cuius numerator ac denominator caſu $x = 0$ evaneſcunt. Ponatur ergo $x = \omega$, & cum ſit

$$e^{\pi\omega} = 1 + \pi\omega + \tfrac{1}{2}\pi^2\omega^2 + \tfrac{1}{6}\pi^3\omega^3 + \&c.$$

formula propoſita tranſmutatur in hanc:

$$\frac{\pi^2\omega + \frac{1}{2}\pi^3\omega^2 + \frac{1}{6}\pi^4\omega^3 + \&c.}{8\omega + 4\pi\omega^2 + 2\pi^2\omega^3 + \&c.}$$

quae poſito ω infinite parvo ſtatim dat $\frac{1}{8}\pi^2$, qui eſt valor quaeſitus expreſſionis propoſitae caſu $x = 0$. At vero expreſſio

ſio propoſita $\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}$, exhibet ſummam huius ſeriei $\frac{1}{1+xx} + \frac{1}{9+xx} + \frac{1}{25+xx} + \frac{1}{49+xx} +$ &c. cuius ſumma poſito $x = 0$ utique fit $= \frac{1}{8}\pi^2$.

EXEMPLUM IV.

Quaeratur valor huius expreſſionis $\frac{1}{2xx} - \frac{\pi}{2x\,tang\,\pi x}$ *caſu* $x = 0$.

Formula haec propoſita $\frac{1}{2xx} - \frac{\pi}{2x\,\mathrm{tang}\,\pi x}$ exprimit ſummam huius ſeriei infinitae $\frac{1}{1-xx} + \frac{1}{4-xx} + \frac{1}{9-xx} + \frac{1}{16-xx} +$ &c. Si igitur ponatur $x = 0$, prodire debet ſumma ſeriei

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ &c. quae eſt $= \frac{1}{6}\pi\pi$.

Quoniam eſt $\mathrm{tang}\,\pi x = \frac{\mathrm{ſin}\,\pi x}{\mathrm{coſ}\,\pi x}$, expreſſio propoſita induet hanc formam: $\frac{1}{2xx} - \frac{\pi\,\mathrm{coſ}\,\pi x}{2x\,\mathrm{ſin}\,\pi x} = \frac{\mathrm{ſin}\,\pi x - \pi x\,\mathrm{coſ}\,\pi x}{2xx\,\mathrm{ſin}\,\pi x}$ cuius numerator ac denominator evaneſcit poſito $x = 0$. Ponatur ergo $x = \omega$ & cum ſit

$\mathrm{ſin}\,\pi x = \pi\omega - \frac{1}{6}\pi^3\omega^3 +$ &c.

$\mathrm{coſ}\,\pi x = 1 - \frac{1}{2}\pi^2\omega^2 +$ &c. expreſſio propoſita fiet:

$$\frac{\pi\omega - \frac{1}{6}\pi^3\omega^3 + \&c. - \pi\omega + \frac{1}{2}\pi^3\omega^3 - \&c.}{2\pi\omega^3 - \frac{1}{3}\pi^3\omega^5 + \&c.} = \frac{\frac{1}{3}\pi^3\omega^3 - \&c.}{2\pi\,\omega^3 - \&c.}$$

quae ob ω infinite parvum dat $\frac{1}{6}\pi^2$.

EXEMPLUM V.

Cum ſit ſumma huius ſeriei infinitae

$$\frac{1}{1-xx} + \frac{1}{9-xx} + \frac{1}{25-xx} + \frac{1}{49-xx} + \&c. = \frac{\pi\,ſin\,\frac{1}{2}\pi x}{4x\,coſ\,\frac{1}{2}\pi x}$$

invenire eius ſummam, ſi fuerit $x = 0$.

Quia eſt $\mathrm{ſin}\,\frac{1}{2}\pi x = \frac{1}{2}\pi x - \frac{1}{48}\pi^3 x^3 +$ &c. & $\mathrm{coſ}\,\frac{1}{2}\pi x = 1 - \frac{1}{8}\pi^2 x^2 +$ &c. erit expreſſio propoſita

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$$=\frac{\frac{1}{2}\pi^2 x-\frac{1}{48}\pi^4 x^3+\&c.}{4x-\frac{1}{2}\pi^2 x^3+\&c.}=\frac{\frac{1}{2}\pi^2-\frac{1}{48}\pi^4 x^2+\&c.}{4-\frac{1}{2}\pi^2 x^2+\&c.}$$

in qua ſi fiat $x=0$, valor erit manifeſto $=\frac{1}{8}\pi^2$, quam eſſe ſummam ſeriei $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\&c.$ ſupra pluribus modis eſt demonſtratum. Sin autem pro x ſumatur numerus par quicunque, ſumma ſeriei propoſitae ſemper eſt $=0$.

366. In his ſeriebus, quas binis ultimis exemplis tractavimus, aliiſque litteram variabilem x continentibus, ipſi x eiuſmodi valores tribui poſſunt, ut quidam termini in infinitum excreſcant, quibus quidem caſibus ſumma totius ſeriei fiet infinita. Sic ſeries:

$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\&c.$ ſi pro x ponatur numerus quicunque integer, unus perpetuo terminus ob denominatorem evaneſcentem fit infinitus; hancque ob cauſam ipſa ſeriei ſumma infinita evadet. Quodſi autem iſte terminus infinitus ex ſerie tollatur, tum ſumma reliqua ſine dubio erit finita, exprimeturque ſumma priori infinita termino iſto infinito mulctata, hoc modo $\infty-\infty$: quemnam ergo habitura ſit valorem determinatum modo hic expoſito inveniri poterit; id quod clarius ex ſubiunctis exemplis perſpicietur.

EXEMPLUM I.

Invenire ſummam ſeriei

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\&c.$$

caſu $x=1$, & demto termino primo, qui hoc caſu in infinitum augetur.

Quia in genere ſumma eſt $=\frac{1}{2xx}-\frac{\pi}{2x\ \mathrm{tang}\ \pi x}$, erit ſumma quaeſita $=\frac{1}{2xx}-\frac{\pi}{2x\ \mathrm{tang}\ \pi x}-\frac{1}{1-xx}$ poſito $x=1$. Sit $x=1+\omega$, & habebitur pro ſumma quaeſita

$\frac{1}{2(1+2\omega+\omega\omega)} - \frac{\pi}{2(1+\omega)\,\text{tang}\,(\pi+\omega\pi)} + \frac{1}{2\omega+\omega\omega}$. At eſt tang $(\pi+\omega\pi) =$ tang $\omega\pi = \pi\omega + \frac{1}{3}\pi^3\omega^3 +$ &c. Unde cum primus terminus $\frac{1}{2xx}$ poſito $x=1$ determinatum habeat valorem $\frac{1}{2}$, duo reliqui tantum termini ſunt ſpectandi, qui erunt

$$\frac{1}{\omega(2+\omega)} - \frac{\pi}{2\omega(1+\omega)(\pi+\frac{1}{3}\pi^3\omega^2)} = \frac{1}{\omega(2+\omega)} - \frac{1}{\omega(2+2\omega)(1+\frac{1}{3}\pi^2\omega^2)}$$

Si quidem ω ſit infinite parvum, quo caſu etiam terminus $\frac{1}{3}\pi^2\omega^2$ negligi poterit. Proveniet autem $\frac{\omega}{\omega(2+\omega)(2+2\omega)} = \frac{1}{4}$ poſito $\omega=0$, eſtque ergo $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ ſumma ſeriei: $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+$ &c. uti aliunde conſtat.

EXEMPLUM II.

Invenire ſummam ſeriei

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{\&c.}$$

caſu quo pro x *ponitur numerus quicunque integer* n *& demto ex ſerie termino illo* $\frac{1}{nn-xx}$, *qui fit infinitus.*

Summa ergo haec, quae quaeritur, ita erit expreſſa $\frac{1}{2xx} - \frac{\pi}{2x\,\text{tang}\,\pi x} - \frac{1}{nn-xx}$, ſi quidem ſtatuatur $x=n$, quo quidem caſu primus terminus $\frac{1}{2xx}$ abit in $\frac{1}{2nn}$, bini vero reliqui ambo fiunt infiniti. Ponatur ergo $x=n+\omega$, & cum ſit tang $(\pi n+\pi\omega) =$ tang $\pi\omega = \pi\omega$, poſito ω infinite parvo, habebimus pro ſumma quaeſita: $\frac{1}{2nn} - \frac{\pi}{2(n+\omega)\pi\omega} + \frac{1}{2n\omega+\omega\omega}$

ſeu $\frac{1}{2nn} - \frac{1}{\omega(2n+2\omega)} + \frac{1}{\omega(2n+\omega)} = \frac{1}{2nn} + \frac{1}{(2n+2\omega)(2n+\omega)}$

unde

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unde ſi fiat $\omega = 0$, prodibit ſumma quaeſita $= \frac{1}{2nn} + \frac{1}{4nn} = \frac{3}{4nn}$. Quo circa erit

$$\frac{3}{4nn} = \frac{1}{1-nn} + \frac{1}{4-nn} + \frac{1}{9-nn} + \dots + \frac{1}{(n-1)^2-nn}$$
$$+ \frac{1}{(n+1)^2-nn} + \frac{1}{(n+2)^2-nn} + \&c.$$

in infinitum, ſive erit iſtius ſeriei infinitae ſumma:

$$\frac{1}{(n+1)^2-nn} + \frac{1}{(n+2)^2-nn} + \frac{1}{(n+3)^2-nn} + \&c.$$
$$= \frac{3}{4nn} + \frac{1}{nn-1} + \frac{1}{nn-4} + \frac{1}{nn-9} + \dots + \frac{1}{nn-(n-1)^2}.$$

EXEMPLUM III.

Invenire ſummam huius ſeriei

$$\frac{1}{1-xx} + \frac{1}{9-xx} + \frac{1}{25-xx} + \frac{1}{49-xx} + \&c.$$

ſi ponatur $x = 1$, *atque terminus primus* $\frac{1}{1-xx}$,

qui hoc caſu fit infinitus, auferatur.

Cum huius ſeriei ſumma ſit in genere $= \frac{\pi \sin \frac{1}{2}\pi x}{4x \cos \frac{1}{2}\pi x}$ erit ſumma quaeſita $= \frac{\pi \sin \frac{1}{2}\pi x}{4\pi x \cos \frac{1}{2}\pi x} - \frac{1}{1-xx}$, ſi ponatur $x = 1$. Quia vero uterque terminus fit infinitus, ponatur $x = 1 - \omega$, & cum ſit $\sin(\frac{1}{2}\pi - \frac{1}{2}\pi\omega) = \cos\frac{1}{2}\pi\omega = 1 - \frac{1}{8}\pi^2\omega^2$, & $\cos(\frac{1}{2}\pi - \frac{1}{2}\pi\omega) = \sin\frac{1}{2}\pi\omega = \frac{1}{2}\pi\omega$ ob ω infinite parvum, habebitur iſta expreſſio: $\frac{\pi(1-\frac{1}{8}\pi^2\omega^2)}{4(1-\omega)\frac{1}{2}\pi\omega} - \frac{1}{2\omega-\omega\omega} = \frac{1}{\omega(2-2\omega)} - \frac{1}{\omega(2-\omega)}$ quae fit $= \frac{1}{4}$ poſito $\omega = 0$, eſtque propterea

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{24} + \frac{1}{48} + \frac{1}{80} + \frac{1}{120} + \&c.$$

EXEMPLUM IV.

Invenire ſummam ſeriei huius:

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{\&c.}$$ *ſi pro* x *ponatur numerus quicunque integer impar* 2 n − 1 *iſque terminus* $\frac{1}{(2n-1)^2-xx}$ *qui hoc caſu fit infinitus, e medio tollatur.*

Erit ergo ſumma, quae quaeritur, $=\frac{\pi\,\text{ſin}\,\frac{1}{2}\pi x}{4x\,\text{coſ}\,\frac{1}{2}\pi x}-\frac{1}{(2n-1)^2-xx}$ poſito $x=2n-1$. Statuamus ergo $x=2n-1-\omega$, exiſtente ω infinite parvo, fietque $\text{ſin}\,\frac{1}{2}\pi x=\text{ſin}\left(\frac{2n-1}{2}\pi-\frac{1}{2}\pi\omega\right)=\pm\,\text{coſ}\,\frac{1}{2}\pi\omega$, ubi ſignum ſuperius valet, ſi fit n numerus impar, inferius vero ſi ſit par. Simili modo erit $\text{coſ}\,\frac{1}{2}\pi x=\text{coſ}\left(\frac{2n-1}{2}\pi-\frac{1}{2}\pi\omega\right)=\pm\,\text{ſin}\,\frac{1}{2}\pi\omega$; ideoque ſive n ſit par ſive impar, erit $\frac{\text{ſin}\,\frac{1}{2}\pi x}{\text{coſ}\,\frac{1}{2}\pi x}=\frac{1}{\text{tang}\,\frac{1}{2}\pi\omega}=\frac{1}{\frac{1}{2}\pi\omega}$. Hinc ſumma quaeſita ita exprimetur: $\frac{1}{2\omega(2n-1-\omega)}-\frac{1}{\omega[2(2n-1)-\omega]}$, eritque propterea $=\frac{1}{4(2n-1)^2}$. Sic ſi fit $n=2$, erit $\frac{1}{36}=-\frac{1}{8}+\frac{1}{16}+\frac{1}{40}+\frac{1}{72}+\frac{1}{112}+$ &c. cuius ſummationis veritas aliunde conſtat.

CA-

CAPUT XVI.

DE DIFFERENTIATIONE FUNCTIONUM INEXPLICABILIUM.

367.

Functiones inexplicabiles hic voco, quae neque expreſſionibus determinatis, neque per aequationum radices explicari poſſunt; ita ut non ſolum non ſint algebraicae, ſed etiam plerumque incertum ſit, ad quod genus tranſcendentium pertineant. Huiuſmodi functio inexplicabilis eſt $1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{x}$, quae utique ab x pendet, at niſi x ſit numerus integer nullo modo explicari poteſt. Simili modo haec expreſſio $1.2.3.4\ldots x$, erit functio inexplicabilis ipſius x, quoniam ſi x ſit numerus quicunque eius valor non ſolum non algebraice, ſed ne quidem per ullum certum quantitatum tranſcendentium genus exprimi poteſt. Generatim ergo talium functionum inexplicabilium notio ex ſeriebus derivari poteſt. Sit enim propoſita ſeries quaecunque

$$\begin{matrix} 1 & & 2 & & 3 & & 4 & & \ldots & & x \\ A & + & B & + & C & + & D & + & \ldots & + & X \end{matrix}$$

cuius ſumma ſi formula finita exprimi nequeat, praebebit functionem inexplicabilem ipſius x, nempe

$$S = A + B + C + D + \ldots + X.$$

Similiter continua producta ex terminis ſerierum uti

$$P = A\,B\,C\,D \ldots X$$

exhibebunt functiones inexplicabiles ipſius x, quae autem ope logarithmorum ad formam priorem revocari poſſunt, erit enim:

$$lP = lA + lB + lC + lD + \ldots + lX$$

368. Hoc igitur capite methodum explicare conſtitui, huiuſmodi functionum inexplicabilium differentialia inveſtigandi.

di. Quod argumentum, quamvis ad primam huius operis partem, ubi praecepta calculi differentialis ſunt tradita, pertinere videatur; tamen quoniam uberiorem doctrinae ſerierum cognitionem poſtulat, ad quam in hac altera parte pervenire licuit, ordinem naturalem relinquere coacti hoc loco attingamus. Cum autem haec inveſtigatio prorſus ſit nova, neque a quoquam adhuc tractata, tantum abeſt ut hanc calculi differentialis partem abſolvere queamus, ut potius prima tantum eius elementa adumbrare conemur. Praeterea vero nonnullas quaeſtiones proponam, quarum enodatio differentiationem huiuſmodi functionum inexplicabilium requirat, quo ſimul uſus huius tractationis, qui autem in poſterum ſine dubio multo amplior erit, clarius perſpiciatur.

369. Ad huiuſmodi functiones inexplicabiles differentiandas ante omnia neceſſe eſt, ut earum valores inveſtigemus, quos induunt, ſi pro x ponatur $x + \omega$. Sit igitur

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \dots + \overset{x}{X}$$

atque ponatur Σ valor ipſius S, quem recipit, ſi pro x ponatur $x + \omega$, ſitque Z terminus ſeriei reſpondens indicii $x + \omega$. Iam igitur termini, qui reſpondent indicibus $x+1$, $x+2$, $x+3$, &c. indicentur per X', X'', X''', X^{IV}, &c. atque is, qui convenit indici infinito $x + \infty$ per $X^{|\infty|}$. Similique modo termini competentes indicibus $x+\omega+1$, $x+\omega+2$, $x+\omega+3$ &c. indicentur per Z', Z'', Z''', &c. & ſit $Z^{|\infty|}$ terminus reſpondens indici $x + \omega + \infty$. Quibus poſitis erit

$$S' = S + X'$$
$$S'' = S + X' + X''$$
$$S''' = S + X' + X'' + X'''$$
&c.
$$S^{|\infty|} = S + X' + X'' + X''' + \dots + X^{|\infty|}.$$

Simili modo cum etiam Σ ſucceſſive terminis Z', Z'' &c. augeatur, erit

Σ'

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + Z''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z'''$$
&c.
$$\Sigma^{|\infty|} = \Sigma + Z' + Z'' + Z''' + \ldots + Z^{|\infty|}$$

370. Nunc natura ſeriei S, S′, S″, S‴, &c. eſt perpendenda, qualis futura ſit, ſi in infinitum continuetur: quae ſi in infinito cum progreſſione arithmetica confundatur; quod fit ſi termini ſeriei X, X′, X″, X‴, &c. in infinito ad aequalitatem convergant, ita ut differentiae ſeriei S,S′,S″, &c. tandem fiant aequales: hoc caſu quantitates $S^{|\infty|}$, $S^{|\infty+1|}$, $S^{|\infty+2|}$ &c. erunt in arithmetica progreſſione, & cum ſit $\Sigma^{|\infty|} = S^{|\infty+\omega|}$ ob $S^{|\infty+\omega|} = S^{|\infty|} + \omega(S^{|\infty+1|} - S^{|\infty|}) = \omega S^{|\infty+1|} + (1-\omega)S^{|\infty|}$ erit $\Sigma^{|\infty|} = \omega S^{|\infty+1|} + (1-\omega)S^{|\infty|}$. At eſt $S^{|\infty+1|} = S^{|\infty|} + X^{|\infty+1|}$, unde fit $\Sigma^{|\infty|} = S^{|\infty|} + \omega X^{|\infty+1|}$, ex quo obtinebitur haec aequatio

$$\Sigma + Z' + Z'' + Z''' + \ldots\ldots + Z^{|\infty|} =$$
$$S + X' + X'' + X''' + \ldots\ldots + X^{|\infty|} + \omega X^{|\infty+1|}$$

ex qua definitur valor quaeſitus Σ, quem induit functio S, dum in ea $x + \omega$ loco x ſubſtituitur; eritque

$$\Sigma = S + \omega X^{|\infty+1|} + X' + X'' + X''' + \text{\&c. in infinitum}$$
$$- Z' - Z'' - Z''' - \text{\&c. in infinitum}$$

Quare ſi ſeriei A, B, C, D, &c. termini infiniteſimi evaneſcant, terminus $\omega X^{|\infty+1|}$ evaneſcit, & omitti poteſt.

371. Exprimitur ergo valor ipſius Σ per novam ſeriem infinitam, quae exhiberi poteſt, ſi ſeriei A + B + C + &c. habeatur terminus generalis, ex quo valores terminorum Z′, Z″, Z‴, &c. definiri queant. Poſito ergo ω infinite parvo, cum ſit $\Sigma - S$ differentiale functionis S, hoc differentiale dS per ſeriem infinitam exprimetur. Atque ſi nequidem altiores poteſtates ipſius ω negligantur, habebitur differentiale completum functionis huius inexplicabilis S, cuius natura, quo clarius ob oculos ponatur, ſequentibus exemplis hoc negotium illuſtrabimus.

EXEM-

EXEMPLUM I.

Invenire differentiale huius functionis inexplicabilis

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots\ldots + \frac{1}{x}.$$

Quoniam huius ſeriei terminus generalis X eſt $= \frac{1}{x}$ ac propterea

$$X' = \frac{1}{x+1} \qquad Z' = \frac{1}{x+1+\omega}$$
$$X'' = \frac{1}{x+2} \qquad Z'' = \frac{1}{x+2+\omega}$$
$$X''' = \frac{1}{x+3} \qquad Z''' = \frac{1}{x+3+\omega}$$
&c. &c.

ob $X|\infty+1| = \frac{1}{x+\infty+1} = 0$, ſi loco x ponatur $x+\omega$ functio S abibit in Σ, ut ſit

$$\Sigma = S + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \text{ \&c.}$$
$$- \frac{1}{x+1+\omega} - \frac{1}{x+2+\omega} - \frac{1}{x+3+\omega} - \text{ \&c.}$$

ſive binis his terminis in ſingulos colligendis, erit $\Sigma = S$
$$+ \frac{\omega}{(x+1)(x+1+\omega)} + \frac{\omega}{(x+2)(x+2+\omega)} + \frac{\omega}{(x+3)(x+3+\omega)} + \text{ \&c.}$$
ſeu cum ſit
$$\frac{1}{x+1+\omega} = \frac{1}{x+1} - \frac{\omega}{(x+1)^2} + \frac{\omega^2}{(x+1)^3} - \frac{\omega^3}{(x+1)^4} + \text{ \&c.}$$
$$\frac{1}{x+2+\omega} = \frac{1}{x+2} - \frac{\omega}{(x+2)^2} + \frac{\omega^2}{(x+2)^3} - \frac{\omega^3}{(x+2)^4} + \text{ \&c.}$$
&c.

erit ſeriebus ſecundum poteſtates ipſius ω diſpoſitis

$\Sigma =$

$$\Sigma = S + \omega \left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \&c.\right)$$
$$- \omega^2 \left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c.\right)$$
$$+ \omega^3 \left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \&c.\right)$$
$$- \omega^4 \left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \&c.\right) \&c.$$

Poſito ergo dx pro ω obtinebimus functionis propoſitae S differentiale completum

$$dS = dx \left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \&c.\right)$$
$$- dx^2 \left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c.\right)$$
$$+ dx^3 \left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \&c.\right)$$
$$- dx^4 \left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \&c.\right) \&c.$$

EXEMPLUM II.

Invenire differentiale huius functionis inexplicabilis ipſius S

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2x-1}.$$

Quia huius ſeriei terminus generalis eſt $X = \frac{1}{2x-1}$; erit:

$X' = \frac{1}{2x+1}$	$Z' = \frac{1}{2x+1+2\omega}$
$X'' = \frac{1}{2x+3}$	$Z'' = \frac{1}{2x+3+2\omega}$
$X''' = \frac{1}{2x+5}$	$Z''' = \frac{1}{2x+5+2\omega}$
&c.	&c.

ob

ob terminos huius ſeriei infiniteſimos evaneſcentes & aequales, prodibit valor ipſius S, ſi loco x ponatur $x+1$:

$$\Sigma = S + \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \&c.$$
$$- \frac{1}{2x+1+2\omega} - \frac{1}{2x+3+2\omega} - \frac{1}{2x+5+2\omega} - \&c.$$

ſeu

$$\Sigma = S + \frac{2\omega}{(2x+1)(2x+1+2\omega)} + \frac{2\omega}{(2x+3)(2x+3+2\omega)} + \&c.$$

Verum ſi ſinguli termini in ſeries ſecundum dimenſiones ipſius ω reſolvantur, erit:

$$\begin{aligned} \Sigma = S &+ 2\omega \left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c.\right) \\ &- 4\omega^2 \left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c.\right) \\ &+ 8\omega^3 \left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \&c.\right) \\ &- 16\omega^4 \left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \&c.\right) \\ &\&c. \end{aligned}$$

Ponatur nunc dx pro ω, atque prodibit differentiale completum functionis inexplicabilis S propoſitae:

$$\begin{aligned} dS = &\ 2dx \left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c.\right) \\ &- 4dx^2 \left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c.\right) \\ &+ 8dx^3 \left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \&c.\right) \\ &- 16dx^4 \left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \&c.\right) \\ &\&c. \end{aligned}$$

EXEM-

EXEMPLUM III.

Invenire differentiale completum functionis huius inexplicabilis ipſius S:

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots\dots + \frac{1}{x^n}.$$

Cum huius ſeriei terminus generalis ſit $= \frac{1}{x^n}$, erunt termini infiniteſimi evaneſcentes & inter ſe aequales. Hincque ob

$$X' = \frac{1}{(x+1)^n} \qquad Z' = \frac{1}{(x+1+\omega)^n}$$
$$X'' = \frac{1}{(x+2)^n} \qquad Z'' = \frac{1}{(x+2+\omega)^n}$$
$$X''' = \frac{1}{(x+3)^n} \qquad Z''' = \frac{1}{(x+3+\omega)^n}$$
&c. &c.

erit:

$$X' - Z' = \frac{n\omega}{(x+1)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+1)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+1)^{n+3}} - \&c.$$
$$X'' - Z'' = \frac{n\omega}{(x+2)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+2)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+2)^{n+3}} - \&c.$$
&c.

ex quibus invenitur:

$$\Sigma - S = n\omega\left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \&c.\right)$$
$$- \frac{n(n+1)}{1.\ 2}\omega^2\left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \&c.\right)$$
$$+ \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\omega^3\left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \&c.\right)$$
&c.

Quare poſito $\omega = dx$ prodibit differentiale completum functionis S quaeſitum:

$$dS = + n\,dx\left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \&c.\right)$$
$$-\frac{n(n+1)}{1.\ 2}dx^2\left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \&c.\right)$$
$$+\frac{n(n+1)(n+2)}{1.\ 2.\ 3}dx^3\left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \&c.\right)$$
&c.

372. Ex his quoque ſummae iſtarum ſerierum interpolari, ſeu valores terminorum ſummatoriorum exhiberi poſſunt, quando numerus terminorum non eſt numerus integer. Si enim ponatur $x = 0$, erit quoque $S = \omega$, atque Σ exprimet ſummam tot terminorum, quot numerus ω continet unitates, etiamſi iſte numerus ω non ſit integer. Ita in exemplo primo ſi ponatur

$$\Sigma = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{\omega}$$

erit:

$$\Sigma = \frac{\omega}{1(1+\omega)} + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \&c.$$

ſive

$$\Sigma = \omega\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \&c.\right)$$
$$-\omega^2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+\omega^3\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$
&c.

In exemplo vero tertio erit:

$$\Sigma = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots + \frac{1}{\omega^n}.$$

Valorque ipſius Σ, ſive ω ſit numerus integer ſive fractus, per ſeries ſequenti modo exprimetur: $\Sigma =$

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$$\Sigma = \quad n\omega\left(1 + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}} + \&c.\right)$$
$$- \frac{n(n+1)}{1.\,2}\omega^2\left(1 + \frac{1}{2^{n+2}} + \frac{1}{3^{n+2}} + \frac{1}{4^{n+2}} + \&c.\right)$$
$$+ \frac{n(n+1)(n+2)}{1.\,2.\,3}\omega^3\left(1 + \frac{1}{2^{n+3}} + \frac{1}{3^{n+3}} + \frac{1}{4^{n+3}} + \&c.\right)$$
&c.

373. Haec eadem quoque ad ſeriem generalem accommodari poſſunt, cum enim ſit

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \;.\;\;.\;\;.\; + \overset{x}{X}$$

atque poſito $x + \omega$ loco x, abeat X in Z, & S in Σ, erit:
$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1.\,2\,dx^2} + \frac{\omega^3 d^3X}{1.2.3\,dx^3} + \&c.$ & quia ſimili modo Z', Z'', Z''', &c. per X', X'', X''', &c. exprimuntur, erit:

$$\Sigma = S + \omega X^{|\infty+1|} - \frac{\omega}{dx}\,d.\,[X' + X'' + X''' + X'''' + \&c.]$$
$$- \frac{\omega^2}{1.\,2\,dx^2}\,dd.\,[X' + X'' + X''' + X'''' + \&c.]$$
$$- \frac{\omega^3}{1.\,2.\,3\,dx^3}\,d^3.\,[X' + X'' + X''' + X'''' + \&c.]$$
&c.

& niſi $X^{|\infty+1|}$ ſit $= 0$, hoc modo exprimi poterit, ut conſideratio infiniti tollatur:

$$X^{|\infty+1|} = X' + (X'' - X') + (X''' - X'') + (X'''' - X''') + \&c.$$

eritque ergo:

$$\Sigma = S + \omega X' + \omega[(X'' - X') + (X''' - X'') + (X'''' - X''') + \&c.]$$
$$- \frac{\omega}{dx}\,d.[X' + X'' + X''' + X'''' + X''''' + \&c.]$$
$$- \frac{\omega^2}{2dx^2}\,dd.[X' + X'' + X''' + X'''' + \&c.]$$
$$- \frac{\omega^3}{6dx^3}\,d^3.[X' + X'' + X''' + X'''' + \&c.] \qquad \&c.$$

 Si

Si ergo ponatur $\omega = dx$, orietur differentiale completum ipſius $S = A + B + C + \ldots + X$, ita expreſſum:

$$dS = X'dx + dx[(X'' - X') + (X''' - X'') + (X'''' - X''') + \&c.]$$
$$- d.[X' + X'' + X''' + X'''' + \&c.]$$
$$- \tfrac{1}{2}dd.[X' + X'' + X''' + X'''' + \&c.]$$
$$- \tfrac{1}{6}d^3.[X' + X'' + X''' + X'''' + \&c.] \quad \&c.$$

374. Ponamus eſſe $x = 0$, fiet $X' = A$, $X'' = B$, &c. ideoque $X' + X'' + X''' + \&c.$ erit ſeries infinita cuius terminus generalis eſt $= X$. Formentur deinde ſeries ex his terminis generalibus:

$$\frac{dX}{dx};\ \frac{ddX}{2dx^2};\ \frac{d^3X}{6dx^3};\ \frac{d^4X}{24dx^4};\ \&c.$$

quarum ſerierum in infinitum continuatarum ſummae ſint:

$$\int. X = \mathfrak{A}$$
$$\int.\frac{dX}{dx} = \mathfrak{B}$$
$$\int.\frac{ddX}{2dx^2} = \mathfrak{C}$$
$$\int.\frac{d^3X}{6dx^3} = \mathfrak{D} \quad \&c.$$

& quia poſito $x = 0$, fit quoque $S = 0$, & Σ erit ſumma ſeriei $A + B + C + D + \ldots + Z$ continentis ω terminos; eſt enim Z terminus indicis ω, ſive ω ſit numerus integer ſive fractus. Quare habebitur

$$\Sigma = \omega A + \omega[(B - A) + (C - B) + (D - C) + \&c.]$$
$$- \omega\mathfrak{B} - \omega^2\mathfrak{C} - \omega^3\mathfrak{D} - \omega^4\mathfrak{E} - \&c.$$

ubi prima ſeries praetermitti poteſt, ſi ſeriei propoſitae termini tandem evaneſcant.

375. Scribamus nunc x loco ω, abibitque Σ in S ita ut ſit

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & \ldots & x \\ S = & A & + & B & + & C & + & D & + \ldots + & X \end{matrix}$$

atque idem ipſius S valor iam per ſeriem infinitam exprimetur hoc modo:

S =

$$S = \mathrm{A}x + x[(\mathrm{B}-\mathrm{A}) + (\mathrm{C}-\mathrm{B}) + (\mathrm{D}-\mathrm{C}) + \&c.]$$
$$- \mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \mathfrak{F}x^5 - \&c.$$

cuius valor cum aeque diſtincte exprimatur, ſive x ſit numerus integer ſive fractus, differentialia ipſius S cuiuſque ordinis hinc facile exhiberi poſſunt:

$$\frac{dS}{dx} = \mathrm{A} + (\mathrm{B}-\mathrm{A}) + (\mathrm{C}-\mathrm{B}) + (\mathrm{D}-\mathrm{C}) + \&c.$$
$$- \mathfrak{B} - 2\mathfrak{C}x - 3\mathfrak{D}x^2 - 4\mathfrak{E}x^3 - \&c.$$
$$\frac{ddS}{2dx^2} = -\mathfrak{C} - 3\mathfrak{D}x - 6\mathfrak{E}x^2 - 10\mathfrak{F}x^3 - \&c.$$
$$\frac{d^3S}{6dx^3} = -\mathfrak{D} - 4\mathfrak{E}x - 10\mathfrak{F}x^2 - 20\mathfrak{G}x^3 - \&c.$$
$$\frac{d^4S}{24dx^4} = -\mathfrak{E} - 5\mathfrak{F}x - 15\mathfrak{G}x^2 - \&c.$$

quare cum differentiale completum ſit

$$= dS + \tfrac{1}{2}ddS + \tfrac{1}{6}d^3S + \tfrac{1}{24}d^4S + \&c.$$

erit functionis propoſitae S differentiale completum:

$$dS = \mathrm{A}dx + (\mathrm{B}-\mathrm{A})dx + (\mathrm{C}-\mathrm{B})dx + (\mathrm{D}-\mathrm{C})dx + \&c.$$
$$- \mathfrak{B}dx - \mathfrak{C}(2xdx + dx^2) - \mathfrak{D}(3x^2dx + 3xdx^2 + dx^3)$$
$$- \mathfrak{E}(4x^3dx + 6x^2dx^2 + 4xdx^3 + dx^4) - \&c.$$

376. Hoc ergo modo functionis cuiuſque inexplicabilis S differentiale aſſignari poteſt, ſi ſeriei $\mathrm{A} + \mathrm{B} + \mathrm{C} + \&c.$ termini infiniteſimi vel evaneſcant vel inter ſe ſint aequales. Quodſi enim huius termini infiniteſimi non fuerint $= 0$, tum ſumma ſeriei $\mathfrak{B}$, quae ex termino generali $\frac{d\mathrm{X}}{dx}$ formatur, fiet infinita; at vero cum ſerie $\mathrm{A} + (\mathrm{B}-\mathrm{A}) + (\mathrm{C}-\mathrm{B}) + (\mathrm{D}-\mathrm{C}) +$ &c. coniuncta ſummam finitam conſtituet. At fieri poteſt, ut termini ſeriei $\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \&c.$ ita in infinitum augeantur, ut non ſolum ſeriei $\mathfrak{B}$, ſed etiam ſeriei $\mathfrak{C}$ ſumma

fiat

fiat infinite magna, quo caſu non ſufficit ſeriem $A+(B-A)+(C-B)+$ &c. adieciſſe: ſed quoniam hoc caſu valores infiniteſimi §. 370. conſiderati, nempe $S^{|\omega|}$, $S^{|\omega+1|}$, $S^{|\omega+2|}$, non amplius in arithmetica ſunt progreſſione, uti aſſumſeramus, huius progreſſionis ratio erit habenda. Quemadmodum ergo aſſumſimus, horum terminorum differentias primas eſſe aequales; ita methodum amplius extendemus, ſi horum valorum differentias demum ſecundas, vel tertias, vel ulteriores conſtantes ſtatuamus.

377. Retento ergo eodem ratiocinio, quo §. 369. ſumus uſi, ponamus memoratorum valorum differentias demum ſecundas eſſe conſtantes:

$$S^{|\omega|},\quad S^{|\omega+1|},\quad S^{|\omega+2|};$$

DIFF. I. $\qquad X^{|\omega+1|},\quad X^{|\omega+2|}$

DIFF. II. $\qquad X^{|\omega+2|}-X^{|\omega+1|}$

Hinc erit $\Sigma^{|\omega|}=S^{|\omega+\omega|}=S^{|\omega|}+\omega X^{|\omega+1|}+\frac{\omega(\omega-1)}{1.2}(X^{|\omega+2|}-X^{|\omega+1|})=S^{|\omega|}-\frac{\omega(\omega-3)}{1.2}X^{|\omega+1|}+\frac{\omega(\omega-1)}{1.2}X^{|\omega+2|}$

Quamobrem habebimus hanc aequationem:

$$\Sigma+Z'+Z''+Z'''+\;.\;.\;.\;+Z^{|\omega|}=$$
$$S+X'+X''+X'''+\;.\;.\;.\;+X^{|\omega|}-$$
$$\frac{\omega(\omega-3)}{1.2}X^{|\omega+1|}+\frac{\omega(\omega-1)}{1.2}X^{|\omega+2|},$$

ex qua elicitur:

$$\Sigma=S+X'+X''+X'''+X''''+\text{\&c. in infinitum}$$
$$-Z'-Z''-Z'''-Z''''-\text{\&c. in infinitum}$$
$$+\omega X^{|\omega+1|}+\frac{\omega(\omega-1)}{1.2}(X^{|\omega+2|}-X^{|\omega+1|}).$$

Termini autem iſti infiniteſimi ita repraeſentari poterunt, ut ſit

$\Sigma =$

$$\Sigma = S + X' + X'' + X''' + X'''' + \&c.$$
$$\quad - Z' - Z'' - Z''' - Z'''' - \&c.$$
$$+ \omega X' + \omega \left\{ \begin{array}{l} + X'' + X''' + X'''' + X''''' + \&c. \\ - X' - X'' - X''' - X'''' - \&c. \end{array} \right.$$
$$\begin{array}{l} + \frac{\omega(\omega - 1)}{1.\ 2} X'' \\ - \frac{\omega(\omega - 1)}{1.\ 2} X' \end{array} + \frac{\omega(\omega - 1)}{1.\ 2} \left\{ \begin{array}{l} + X''' + X^{iv} + X^{v} + \&c. \\ - 2X'' - 2X''' - 2X^{iv} - \&c. \\ + X' + X'' + X''' + \&c. \end{array} \right\}$$

unde ſimul lex patet, qua haec expreſſio erit comparata, ſi differentiae demum tertiae vel quartae vel ulteriores fuerint conſtantes.

378. Cum igitur ſit, ut ſupra demonſtravimus:

$$Z = X + \frac{\omega dX}{1dx} + \frac{\omega^2 ddX}{1.2dx^2} + \frac{\omega^3 d^3X}{1.2.3dx^3} + \&c.$$

ſi loco Z', Z'', Z''', &c. valores hinc oriundos ſubſtituamus, erit valor ipſius S, ſi loco x ſcribatur $x + \omega$, ſequens:

$$\Sigma = S + \omega X' + \omega \left\{ \begin{array}{l} + X'' + X''' + X^{iv} + X^{v} + \&c. \\ - X' - X'' - X''' - X^{iv} - \&c. \end{array} \right\}$$
$$\begin{array}{l} + \frac{\omega(\omega - 1)}{1.\ 2} X'' \\ - \frac{\omega(\omega - 1)}{1.\ 2} X' \end{array} + \frac{\omega(\omega - 1)}{1.\ 2} \left\{ \begin{array}{l} + X''' + X^{iv} + X^{v} + X^{vi} + \&c. \\ - 2X'' - 2X''' - 2X^{iv} - 2X^{v} - \&c. \\ + X' + X'' + X''' + X^{iv} + \&c. \end{array} \right\}$$
$$- \frac{\omega}{dx} d. (X' + X'' + X''' + X'''' + \&c.)$$
$$- \frac{\omega^2}{2dx^2} d^2. (X' + X'' + X''' + X'''' + \&c.)$$
$$- \frac{\omega^3}{6dx^3} d^3. (X' + X'' + X''' + X'''' + \&c.)$$
$$\&c.$$

Si ergo loco ω ponatur dx, prodibit differentiale completum functionis inexplicabilis propoſitae S; ſcilicet

dS

$$dS = X'dx + dx\left\{\begin{array}{l} +X''+X'''+X^{\text{iv}}+X^{\text{v}}+\&c. \\ -X'-X''-X'''-X^{\text{iv}}-\&c.\end{array}\right\}$$

$$-X''\frac{dx(1-dx)}{1.\;2} + X'\frac{dx(1-dx)}{1.\;2} - \frac{dx(1-dx)}{1.\;2}\left\{\begin{array}{l} +X'''+X^{\text{iv}}+X^{\text{v}}+X^{\text{vi}}+\&c. \\ -2X''-2X'''-2X^{\text{iv}}-2X^{\text{v}}-\&c. \\ +X'+X''+X'''+X^{\text{iv}}+\&c.\end{array}\right\}$$

$$+X'''\frac{dx(1-dx)(2-dx)}{1.\;2.\;3} - 2X''\frac{dx(1-dx)(2-dx)}{1.\;2.\;3} + X'\frac{dx(1-dx)(2-dx)}{1.\;2.\;3} + \frac{dx(1-dx)(2-dx)}{1.\;2.\;3}\left\{\begin{array}{l} +X^{\text{iv}}+X^{\text{v}}+\&c. \\ -3X'''-3X^{\text{iv}}-\&c. \\ +3X''+3X'''+\&c. \\ -X'-X''-\&c.\end{array}\right\}$$

&c.

$$\begin{aligned} &- d.\,(X'+X''+X'''+X^{\text{iv}}+X^{\text{v}}+\&c.) \\ &-\tfrac{1}{2}dd.\,(X'+X''+X'''+X^{\text{iv}}+X^{\text{v}}+\&c.) \\ &-\tfrac{1}{6}d^3.\,(X'+X''+X'''+X^{\text{iv}}+X^{\text{v}}+\&c.) \\ &-\tfrac{1}{24}d^4.\,(X'+X''+X'''+X^{\text{iv}}+X^{\text{v}}+\&c.) \end{aligned}$$

&c.

quae expreſſio latiſſime patet, & quotaecunque demum differentiae fuerint conſtantes, differentiale quaeſitum exhibet. Accommodata enim eſt haec formula ad differentias conſtantes, & ſimul lex patet, ſi forte ulterius progredi neceſſe ſit.

379. Quod ſi ſeries $A + B + C + D +$ &c. ex qua formatur functio inexplicabilis

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$$

ita fuerit comparata, ut eius termini infiniteſimi evaneſcant, tum uti iam notavimus erit:

$$\begin{aligned} dS = &- d.\,(X'+X''+X'''+X^{\text{iv}}+\&c.) \\ &-\tfrac{1}{2}dd.\,(X'+X''+X'''+X^{\text{iv}}+\&c.) \\ &-\tfrac{1}{6}d^3.\,(X'+X''+X'''+X^{\text{iv}}+\&c.) \\ &-\tfrac{1}{24}d^4.\,(X'+X''+X'''+X^{\text{iv}}+\&c.) \end{aligned}$$

&c.

Sin

Sin autem illius ſeriei termini infiniteſimi non ſint $=0$, ſed tamen differentias habeant evaneſcentes, tum ad iſtam expreſſionem inſuper addi debet

$$dx \left\{ \begin{array}{l} X' + X'' + X''' + X^{\text{IV}} + X^{\text{V}} + \&c. \\ \quad\;\, - X' - X'' - X''' - X^{\text{IV}} - \&c. \end{array} \right\}$$

Verum ſi terminorum infinitesimorum huius ſeriei $A + B + C + D +$ &c. differentiae demum ſecundae evaneſcant, tum praeterea adiici oportet:

$$\frac{dx(dx-1)}{1.\ 2} \left\{ \begin{array}{rrrrl} +X'' & + X''' & + X^{\text{IV}} & + X^{\text{V}} & + \&c. \\ -X' & - 2X'' & - 2X''' & - 2X^{\text{IV}} & - \&c. \\ & + X' & + X'' & + X''' & + \&c. \end{array} \right\}$$

Atque ſi memoratorum terminorum infiniteſimorum differentiae demum tertiae fuerint evaneſcentes, tum praeter has iam exhibitas expreſſiones inſuper addi debet.

$$\frac{dx(dx-1)(dx-2)}{1.\ 2.\ 3} \left\{ \begin{array}{rrrrl} X''' & + X^{\text{IV}} & + X^{\text{V}} & + X^{\text{VI}} & + \&c. \\ -2X'' & - 3X''' & - 3X^{\text{IV}} & - 3X^{\text{V}} & - \&c. \\ + X' & + 3X'' & + 3X''' & + 3X^{\text{IV}} & + \&c. \\ & - X' & - X'' & - X''' & - \&c. \end{array} \right\}$$

Sicque porro expreſſiones inſuper addendae erunt comparatae, ſi ulteriores demum differentiae terminorum infiniteſimorum ſeriei $A + B + C + D +$ &c. evaneſcant. Hincque adeo quaecunque ſeries aſſumatur, dummodo eius termini infiniteſimi tandem ad differentias evaneſcentes perducantur, functionis inexplicabilis ex ea formatae differentiale definiri poterit.

380. Si ponatur $x = 0$; fiet $X' = A$; $X'' = B$, $X''' = C$ &c. Quare uti $A + B + C + D +$ &c. eſt ſeries, cuius terminus generalis eſt X, ſi ex terminis generalibus $\frac{dX}{dx}$; $\frac{ddX}{2dx^2}$; $\frac{d^3X}{6dx^3}$; $\frac{d^4X}{24dx^4}$; &c. ſimili modo formentur ſeries infinitae, earumque ſummae denotentur per litteras: $\mathfrak{B}$; $\mathfrak{C}$; $\mathfrak{D}$; $\mathfrak{E}$; &c. reſpective. Summa ω terminorum ſeriei $A + B + C + D +$ &c. ita exprimetur, ut perinde ſit, ſive ω ſit numerus integer ſive ſecus. Scribamus ergo x pro ω, ut ſit:

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} \ldots + \overset{x}{X}$$

atque ſi huius ſeriei termini infiniteſimi evaneſcant, erit

$$S = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

At ſi termini infiniteſimi differentias ſaltem primas habeant conſtantes, tum ad hunc valorem inſuper addi debet hic:

$$x \left\{ A \begin{array}{l} + B + C + D + E + \&c. \\ - A - B - C - D - \&c. \end{array} \right\}$$

ſin autem illorum terminorum infiniteſimorum differentiae demum ſecundae evaneſcant, tum praeterea addi debet:

$$\frac{x(x-1)}{1.\ 2} \begin{array}{l} \left\{ +B \begin{array}{l} + C + D + E + F + \&c. \\ - 2B - 2C - 2D - 2E - \&c. \end{array} \right\} \\ \left\{ -A \;\; + A + B + C + D + \&c. \right\} \end{array}$$

Si differentiae demum tertiae fuerint evaneſcentes, tum inſuper adiici debet haec ſeries infinita:

$$\frac{x(x-1)(x-2)}{1.\ 2.\ 3} \begin{array}{l} \left\{ \begin{array}{l} + C \\ - 2B \end{array} \begin{array}{l} + D + E + F + G + \&c. \\ - 3C - 3D - 3E - 3F - \&c. \end{array} \right\} \\ \left\{ + A \begin{array}{l} + 3B + 3C + 3D + 3E + \&c. \\ - A - B - C - D - \&c. \end{array} \right\} \end{array}$$

&c.

381. Accommodemus haec quoque ad alterum functionum inexplicabilium genus, quae conſtant continuo producto terminorum aliquot ſeriei popoſitae $A + B + C + D +$ &c. ſitque

$$S = \overset{1}{A} \; \overset{2}{B} \; \overset{3}{C} \; \overset{4}{D} \ldots \overset{x}{X}$$

& quaeratur primo valor Σ, in quem S tranſmutatur, ſi loco x ſcribatur $x + \omega$; ponamus autem ut ante eſſe Z terminum ſeriei $A + B + C + D +$ &c. cuius index ſit $= x + \omega$, uti X reſpondet indici x. Quo ergo hunc caſum ad praecedentem reducamus ſumamus logarithmos, eritque

$$lS = lA + lB + lC + lD + \ldots + lX$$

Quod ſi iam huius ſeriei termini infiniteſimi evaneſcant, erit eandem methodum, qua ante uſi ſumus, adhibendo

$l\Sigma$

$$l\Sigma = lS + lX' + lX'' + lX''' + \&c. \\ - lZ' - lZ'' - lZ''' - \&c.$$

hincque ad numeros regrediendo erit

$$\Sigma = S . \frac{X'}{Z'} . \frac{X''}{Z''} . \frac{X'''}{Z'''} . \frac{X''''}{Z''''}, \&c.$$

quae ergo expreſſio valet, ſi ſeriei A, B, C, D, &c. termini infiniteſimi unitati aequentur. Sin autem logarithmi terminorum infiniteſimorum huius ſeriei non evaneſcant, at tamen differentias habeant evaneſcentes; tum ad illam ſeriem, quam pro $l\Sigma$ invenimus, inſuper addi debet haec ſeries

$$\omega lX' + \omega \left(l\frac{X''}{X'} + l\frac{X'''}{X''} + l\frac{X''''}{X'''} + \&c. \right)$$

ſicque numeris ſumendis habebitur

$$\Sigma = SX^{I\omega} . \frac{X^{II\omega} . X^{I(1-\omega)}}{Z'} . \frac{X^{III\omega} . X^{II(1-\omega)}}{Z''} . \frac{X^{IV\omega} . X^{III(1-\omega)}}{Z'''} . \&c.$$

382. Quodſi ergo ponamus $x = 0$, quo caſu fit $S = 1$ & $X' = A$, $X'' = B$, $X''' = C$, &c. Σ denotabit productum ω terminorum huius ſeriei A, B, C, D, &c. Si igitur pro ω ſcribamus x, ut Σ obtineat valorem, quem ante ipſi S tribueramus, ita ut ſit

$$\begin{array}{llllllllll} & 1 & 2 & 3 & 4 & . & . & . & . & x \\ S = & A. & B. & C. & D. & . & . & . & . & X \end{array}$$

quia nunc Z', Z'', Z''', &c. abeunt in X', X'', X''' &c. ſi logarithmi terminorum infiniteſimorum iſtius ſeriei A, B, C, D, E, &c. evaneſcant, exprimetur S hoc modo

$$S = \frac{A}{X'} . \frac{B}{X''} . \frac{C}{X'''} . \frac{D}{X^{IV}} . \frac{E}{X^{V}}, \&c.$$

Sin autem differentiae demum logarithmorum terminorum infiniteſimorum ſeriei A, B, C, D, &c. evaneſcant, tum iſta functio S ſequenti modo exprimetur, ut ſit:

$$S = A^x . \frac{B^x A^{1-x}}{X'} . \frac{C^x B^{1-x}}{X''} . \frac{D^x C^{1-x}}{X'''} . \frac{E^x D^{1-x}}{X^{IV}} . \&c.$$

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ſi illorum logarithmorum differentiae ſecundae demum ſint evaneſcentes, ex praecedentibus facile colligitur, cuiuſmodi factores inſuper addi debeant; quem caſum, cum vix occurrere ſoleat, hic praetermittamus. Ceterum uſum harum expreſſionum in interpolationis negotio capite ſequente oſtendam.

383. Hic igitur cum differentiatio huiuſmodi functionum inexplicabilium potiſſimum ſit propoſita: inveſtigemus differentiale huius functionis

$$S = A.\,B.\,C.\,D.\,\ldots\ldots\,X$$

Ad hoc reſumamus aequationem ante inventam

$$l\Sigma = lS + lX' + lX'' + lX''' + \&c. - lZ' - lZ'' - lZ''' - \&c.$$

& cum lZ oriatur ex lX, ſi loco x ponatur $x+\omega$, erit

$$lZ = lX + \frac{\omega}{dx}d.lX + \frac{\omega^2}{2dx^2}dd.lX + \frac{\omega^3}{6dx^3}d^3.lX + \&c.$$

quibus valoribus, pro lZ', lZ'', lZ''', &c. ſubſtitutis habebitur

$$\begin{aligned} l\Sigma = lS &- \frac{\omega}{dx}\,d.\,(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.) \\ &- \frac{\omega^2}{2dx^2}dd.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.) \\ &- \frac{\omega^3}{6dx^3}d^3.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.) \\ &\quad \&c. \end{aligned}$$

Ponatur nunc $\omega = dx$, fietque $l\Sigma = lS + d.lS$, ideoque erit

$$\begin{aligned} \frac{dS}{S} = &- d.\,(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.) \\ &- \tfrac{1}{2}dd.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.) \\ &- \tfrac{1}{6}d^3.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.) \\ &\quad \&c. \end{aligned}$$

quae formula valet, ſi logarithmi terminorum infiniteſimorum ſeriei A, B, C, D, &c. evaneſcant; ſin autem ipſi non evaneſcant, attamen differentias habeant evaneſcentes, tum ad praecedentem differentialis completi expreſſionem inſuper addi debet haec ſeries:

$$dx\,lX' + dx\left(l\frac{X''}{X'} + l\frac{X'''}{X''} + l\frac{X''''}{X'''} + \&c.\right)$$

ut obtineatur differentiale completum.

384. Idem adhuc alio modo praeſtari poteſt. Ponatur $x = 0$, quo caſu abit lS in o. Tum formentur ſeries, quarum termini generales ſint:

$$lX;\ \frac{d.lX}{dx};\ \frac{dd.lX}{2dx^2};\ \frac{d^3.lX}{6dx^3};\ \&c.$$

harumque ſerierum infinitarum ſummae ſint reſpective: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. Scribatur x pro ω, ut ſit $\Sigma = S$, eritque

$$lS = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

ſi quidem logarithmi terminorum infiniteſimorum ſeriei A, B, C, D, &c. cuius terminus generalis eſt X, evaneſcant: at ſi horum logarithmorum differentiae demum evaneſcant erit:

$$lS = x\,lA + x\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$
$$-\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

Hincque adeo differentiale ipſius lS erit:

$$\frac{dS}{S} = dx\,lA + dx\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$
$$-\mathfrak{B}dx - 2\mathfrak{C}x\,dx - 3\mathfrak{D}x^2dx - 4\mathfrak{E}x^3dx - \&c.$$

At ſi differentiale completum deſideretur, erit id:

$$\frac{dS}{S} = dx\,lA + dx\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$
$$-\mathfrak{B}dx - \mathfrak{C}(2xdx + dx^2) - \mathfrak{D}(3xxdx + 3xdx^2 + dx^3) - \&c.$$

Ad quarum formularum uſum oſtendendum ſequentia exempla adiicimus, quae utroque modo reſolvemus.

EXEMPLUM I.

Invenire differentiale huius functionis inexplicabilis:

$$S = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\cdots\frac{2x-1}{2x}.$$

Hic ante omnia notandum eſt, terminos infiniteſimos horum

rum factorum abire in unitates, ideoque eorum logarithmos evaneſcere. Cum igitur ſit $X = \frac{2x-1}{2x}$, erit

$$X' = \frac{2x+1}{2x+2};\ X'' = \frac{2x+3}{2x+4};\ X''' = \frac{2x+5}{2x+6};\ \&c.$$

& generaliter $X^{|n|} = \frac{2x+2n-1}{2x+2n}$; unde erit:

$$lX^{|n|} = l(2x+2n-1) - l(2x+2n)$$

$$d.lX^{|n|} = \frac{2dx}{2x+2n-1} - \frac{2dx}{2x+2n}$$

$$dd.lX^{|n|} = -\frac{4dx^2}{(2x+2n-1)^2} + \frac{4dx^2}{(2x+2n)^2}$$

$$d^3.lX^{|n|} = +\frac{2.\,2.\,4dx^3}{(2x+2n-1)^3} - \frac{2.\,2.\,4dx^3}{(2x+2n)^3}$$

$$d^4.lX^{|n|} = -\frac{2.\,2.\,4.\,6dx^4}{(2x+2n-1)^4} + \frac{2.\,2.\,4.\,6dx^4}{(2x+2n)^4}$$

&c.

unde erit differentiale completum:

$$\frac{dS}{S} = -2dx\left\{\begin{matrix} \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \&c. \\ -\frac{1}{2x+2} - \frac{1}{2x+4} - \frac{1}{2x+6} - \&c. \end{matrix}\right\}$$

$$+\tfrac{4}{2}dx^2\left\{\begin{matrix} \frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c. \\ -\frac{1}{(2x+2)^2} - \frac{1}{(2x+4)^2} - \frac{1}{(2x+6)^2} - \&c. \end{matrix}\right\}$$

$$-\tfrac{8}{3}dx^3\left\{\begin{matrix} \frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c. \\ -\frac{1}{(2x+2)^3} - \frac{1}{(2x+4)^3} - \frac{1}{(2x+6)^3} - \&c. \end{matrix}\right\}$$

&c.

Quod

Quod ſi autem tantum differentiale primum quaeratur erit id:

$$\frac{dS}{S} = -2dx \text{ in }$$

$$\left(\frac{1}{(2x+1)(2x+2)} + \frac{1}{(2x+3)(2x+4)} + \frac{1}{(2x+5)(2x+6)} + \&c.\right)$$

quod idem altera methodo §. 384. tradita ita inveſtigatur. Cum ſit $lX = l\frac{2x-1}{2x}$, erit $\frac{d.\,lX}{dx} = \frac{2}{2x-1} - \frac{1}{x}$;

$$\frac{dd.\,lX}{2dx^2} = -\frac{2}{(2x-1)^2} + \frac{1}{2xx};\ \frac{d^3.\,lX}{6dx^3} = +\frac{8}{3(2x-1)^3} - \frac{1}{3x^3}\ \&c.$$

ideoque fiet

$$\mathfrak{A} = l\frac{1}{2} + l\frac{3}{4} + l\frac{5}{6} + l\frac{7}{8} + \&c.$$

$$\mathfrak{B} = \begin{cases} +\frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \&c. \\ -\frac{2}{2} - \frac{2}{4} - \frac{2}{6} - \frac{2}{8} - \frac{2}{10} - \&c. \end{cases} = 2\,l\,2$$

$$\mathfrak{C} = -\frac{4}{2}\begin{cases} \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. \\ -\frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \&c. \end{cases}$$

$$\mathfrak{D} = \frac{8}{3}\begin{cases} \frac{1}{1} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \&c. \\ -\frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{6^3} - \frac{1}{8^3} - \&c. \end{cases}\ \&c.$$

ſive erit;

$$\mathfrak{B} = +\frac{2}{1}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.\right)$$

$$\mathfrak{C} = -\frac{4}{2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \&c.\right)$$

$\mathfrak{D} =$

$$\mathfrak{D} = +\frac{8}{3}\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \&c.\right)$$

$$\mathfrak{E} = -\frac{16}{4}\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c.\right)$$

&c.

Quibus valoribus inventis ſubſtitutis erit:

$$\frac{dS}{S} = -2\,dx\left(1 - \frac{1}{2} \quad \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.\right)$$

$$+4x\,dx\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \&c.\right)$$

$$-8x^2 dx\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \&c.\right)$$

$$+16x^3 dx\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c.\right)$$

&c.

Si igitur fit $x = 0$, quo caſu fit $lS = 0$ & $S = 1$, erit $dS = -2\,dx\,l2$.

EXEMPLUM II.

Invenire differentiale huius functionis inexplicabilis:

$$S = 1 .\ 2 .\ 3 .\ 4 \ldots\ldots x$$

Huius ſeriei 1, 2, 3, 4, &c. termini in infinitum ita creſcunt, ut logarithmorum differentiae evaneſcant: eſt enim $l(\infty + 1) - l\,\infty = l\left(1 + \frac{1}{\infty}\right) = \frac{1}{\infty} = 0$. Cum igitur ſit $X = x$ erit $X' = x + 1$; $X'' = x + 2$; $X''' = x + 3$; &c. porro autem ob $lX = lx$ fiet $d.\,lX = \frac{dx}{x}$; $dd.\,lX = -\frac{dx^2}{x^2}$; $d^3.\,lX = \frac{2dx^3}{x^3}$; $d^4.\,lX = -\frac{2.\,3dx^4}{x^4}$; &c. unde ſi logarithmi ultimi evaneſcerent, foret

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$$\frac{dS}{S} = -dx\left(\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\frac{1}{x+4}+\&c.\right)$$
$$+\frac{dx^2}{2}\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+4)^2}+\&c.\right)$$
$$-\frac{dx^3}{3}\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\frac{1}{(x+4)^3}+\&c.\right)$$
&c.

At cum differentiae demum logarithmorum evaneſcant, inſuper addi debet haec expreſſio:

$$dx\,l(x+1)+dx\left(l\frac{x+2}{x+1}+l\frac{x+3}{x+2}+l\frac{x+4}{x+3}+l\frac{x+5}{x+4}+\&c.\right)$$

Quia vero eſt:

$$l\frac{x+2}{x+1}=\frac{1}{x+1}-\frac{1}{2(x+1)^2}+\frac{1}{3(x+1)^3}-\frac{1}{4(x+1)^4}+\&c.$$
$$l\frac{x+3}{x+2}=\frac{1}{x+2}-\frac{1}{2(x+2)^2}+\frac{1}{3(x+2)^3}-\frac{1}{4(x+2)^4}+\&c.$$
&c.

erit verum differentiale completum:

$$\frac{dS}{S}=dx\,l(x+1)$$
$$-\tfrac{1}{2}(dx-dx^2)\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c.\right)$$
$$+\tfrac{1}{3}(dx-dx^3)\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\&c.\right)$$
$$-\tfrac{1}{4}(dx-dx^4)\left(\frac{1}{(x+1)^4}+\frac{1}{(x+2)^4}+\frac{1}{(x+3)^4}+\&c.\right)$$
$$+\tfrac{1}{5}(dx-dx^5)\left(\frac{1}{(x+1)^5}+\frac{1}{(x+2)^5}+\frac{1}{(x+3)^5}+\&c.\right)$$
&c.

Sin autem altero modo differentiale hoc exprimere velimus, quia eſt

$$lX = lx;\ \frac{d.lX}{dx} = \frac{1}{x};\ \frac{dd.lX}{2dx^2} = -\frac{1}{2x^2};$$

$$\frac{d^3.lX}{6dx^3} = \frac{1}{3x^3};\ \frac{d^4.lX}{24dx^4} = -\frac{1}{4x^4};\ \&c.$$

habebuntur ſequentes ſeries:

$$\mathfrak{A} = l1 + l2 + l3 + l4 + l5 + \&c.$$

$$\mathfrak{B} = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.\right)$$

$$\mathfrak{C} = -\tfrac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$

$$\mathfrak{D} = \tfrac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$

$$\mathfrak{E} = -\tfrac{1}{4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$

&c.

Hinc ob $lA = l1 = 0$, fiet ex §. 384:

$$\begin{aligned} lS = {} & n\left(l\frac{2}{1} + l\frac{3}{2} + l\frac{4}{3} + l\frac{5}{4} + \&c.\right) \\ & - n\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.\right) \\ & + \tfrac{1}{2}n^2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right) \\ & - \tfrac{1}{3}n^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c.\right) \\ & + \tfrac{1}{4}n^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c.\right) \end{aligned}$$

&c.

Binae autem primae ſeries, per quas n eſt multiplicatum, etiamſi utraque habeat ſummam infinitam, tamen ambae ſimul ſummam habent finitam. Si enim utriuſque n termini capiantur, prodibit:

$l(n$

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$$l(n+1) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n}.$$

At ſupra §. 142. invenimus eſſe

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \text{Conſt.} + l\,n$$
$$+ \frac{1}{2n} - \frac{\mathfrak{A}}{2n^2} + \frac{\mathfrak{B}}{4n^4} - \&c.$$

haecque conſtans prodit $= 0,5772156649015325$. Quodſi ergo ponatur $n = \infty$, erit:

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{\infty} = \text{Conſt.} + l\,\infty$, unde binarum illarum ſerierum in infinitum continuatarum valor erit $= l(\infty + 1) - \text{Conſt.} - l\,\infty = -\text{Conſt.}$ Ex quo erit:

$$\begin{aligned}
lS = {} & -x \cdot 0,5772156649015325 \\
& + \tfrac{1}{2}xx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right) \\
& - \tfrac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right) \\
& + \tfrac{1}{4}x^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right) \\
& \&c.
\end{aligned}$$

unde differentialia cuiuſque ordinis facile reperiuntur. Erit enim:

$$\begin{aligned}
\frac{dS}{S} = {} & -dx \cdot 0,5772156649015325 \\
& + x\,dx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right) \\
& - x^2dx\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right) \\
& + x^3dx\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right) \\
& \&c.
\end{aligned}$$

At

At ſi hae ſeries in unam colligantur erit:

$$\frac{dS}{S} = -dx.\,0,5772156649015325 + \frac{x\,dx}{1(1+x)} + \frac{x\,dx}{2(2+x)} + \frac{x\,dx}{3(3+x)} + \frac{x\,dx}{4(4+x)} + \&c.$$

Quare ſi ſit $x=0$, fiet:

$$\frac{dS}{S} = -dx.\,0,5772156649015325$$

Ex priori vero expreſſione hoc caſu erit:

$$\frac{dS}{S} = -\tfrac{1}{2}dx\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.\right)$$
$$+\tfrac{1}{3}dx\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.\right)$$
$$-\tfrac{1}{4}dx\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.\right)$$
$$+\tfrac{1}{5}dx\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c.\right)$$
&c.

385. Hinc ergo etiam huiuſmodi functionum inexplicabilium differentialia quovis caſu ſpeciali exhiberi poſſunt, propterea quod hic differentialia completa eruimus. Quamobrem ſi tales functiones ingrediantur in expreſſiones, quae indeterminatae videntur, cuiuſmodi capite praecedente tractavimus; valores eadem methodo definiri poterunt, uti ex adiunctis exemplis intelligetur.

EXEMPLUM I.

Determinare valorem huius expreſſionis:

$$\frac{1+\frac{1}{2}+\frac{1}{3}+\;.\;.\;.\;+\frac{1}{x}}{x(x-1)} - \frac{1}{(x-1)(2x-1)}$$

eo caſu, quando ponitur $x=1$.

Ponamus $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x}=S$, erit ex §. 372.

S =

$$S = x\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$
$$- x^2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c.\right)$$
$$+ x^3\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c.\right)$$
$$\&c.$$

ſeu cum ſit quoque $S = +\ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$

$$- \frac{1}{1+x} - \frac{1}{2+x} - \frac{1}{3+x} - \frac{1}{4+x} - \frac{1}{5+x} - \&c.$$

ſi quivis terminus ſuperioris ſeriei cum praecedente inferioris combinetur, prodibit:

$$S = 1 + \frac{x-1}{2(1+x)} + \frac{x-1}{3(2+x)} + \frac{x-1}{4(3+x)} + \&c.$$

quae expreſſio, quoniam poni debet $x = 1$ eſt commodior. Sit ergo $x = 1 + \omega$, fietque

$$S = 1 + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \&c.$$

ſive $S = 1 + \omega\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right) = 1 + \mathfrak{B}\omega$

$$- \omega^2\left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right) \qquad - \mathfrak{C}\omega^2$$
$$+ \omega^3\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right) \qquad + \mathfrak{D}\omega^3$$
$$\&c. \qquad\qquad \&c.$$

Tota ergo expreſſio poſito $x = 1 + \omega$ abibit in hanc:

$$\frac{1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \&c.}{\omega(1+\omega)} - \frac{1}{\omega(1+2\omega)} \quad \text{ſeu}$$

$$\frac{\omega + \mathfrak{B}\omega + 2\mathfrak{B}\omega^2 - \mathfrak{C}\omega^2 - \&c.}{\omega(1+\omega)(1+2\omega)} = \frac{1 + \mathfrak{B} + 2\mathfrak{B}\omega - \mathfrak{C}\omega - \&c.}{(1+\omega)(1+2\omega)}$$

Ponatur nunc $\omega = 0$, atque expreſſionis propoſitae valor caſu $x = 1$, erit $= 1 + \mathfrak{B} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$ quae ſeries cum ſit $= \frac{1}{6}\pi^2$, ſequitur valorem quaeſitum eſſe $= \frac{1}{6}\pi^2$.

EXEMPLUM II.

Invenire valorem huius expreſſionis:

$$\frac{2x - xx}{(x-1)^2} + \frac{\pi\pi x}{6(x-1)} - \frac{(2x-1)(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{x})}{x(x-1)^2}$$

caſu quo ponitur $x = 1$.

Ponatur $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{x} = S$, ſtatuaturque $x = 1 + \omega$, fiet ut in exemplo praecedente invenimus: $S = 1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \&c.$ exiſtente

$$\mathfrak{B} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c. = \frac{1}{6}\pi\pi - 1$$

$$\mathfrak{C} = \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.$$

$$\mathfrak{D} = \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.$$

&c.

Poſito ergo $x = 1 + \omega$ expreſſio propoſita induet hanc formam:

$$\frac{1 - \omega\omega}{\omega\omega} + \frac{(1+\mathfrak{B})(1+\omega)}{\omega} - \frac{(1+2\omega)(1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \&c.)}{(1+\omega)\omega^2}$$

quae ad eandem denominationem $\omega^2(1+\omega)$ perducta fit:

$$\frac{\begin{array}{c} 1 + \omega - \omega^2 - \omega^3 + \omega + 2\omega^2 + \omega^3 + \mathfrak{B}\omega(1 + 2\omega + \omega\omega) \\ - 1 - \mathfrak{B}\omega + \mathfrak{C}\omega^2 - \mathfrak{D}\omega^3 - 2\omega - 2\mathfrak{B}\omega^2 + 2\mathfrak{C}\omega^3 \&c. \end{array}}{\omega^2(1+\omega)}$$

quae reducitur ad hanc formam:

$$\frac{\omega^2 + \mathfrak{C}\omega^2 + \mathfrak{B}\omega^3 + 2\mathfrak{C}\omega^3 - \mathfrak{D}\omega^3 \&c.}{\omega^2(1+\omega)}$$

Fiat nunc $\omega = 0$, atque prodibit $1 + \mathfrak{C}$. Quocirca expreſſionis propoſitae valor caſu $x = 1$, erit $= 1 + \mathfrak{C}$, ideoque per

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hanc ſeriem exprimetur: $1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+$ &c. cuius ſumma cum neque per logarithmos, neque per peripheriam circuli π exhiberi poſſit, valor quaeſitus etiamnum alio modo finite aſſignari non poteſt. Ex his ergo duobus exemplis uſus, quem differentiatio functionum inexplicabilium in doctrina ſerierum habere poteſt, ſatis luculenter perſpicitur.

386. In methodo hic tradita functiones inexplicabiles differentiandi aſſumſimus ſeriei A, B, C, D, E, &c. terminos infiniteſimos vel eſſe $=0$, vel differentias tandem evaneſcentes habere; quorum ſi neutrum contingat, iſta methodo uti non licebit. Hancobrem aliam exponam methodum huic conditioni non adſtrictam, quam ſummatio generalis ſerierum ex termino generali petita & ſupra fuſius explicata ſuppeditat. Denotent igitur litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, &c. numeros Bernoullianos §. 122. exhibitos, ſitque functio inexplicabilis propoſita haec: $S=\overset{1}{A}+\overset{2}{B}+\overset{3}{C}+\overset{4}{D}+\ldots+\overset{x}{X}$ & quia ſupra (130.) oſtendimus fore:

$$S=\int Xdx+\tfrac{1}{2}X+\frac{\mathfrak{A}\,dX}{1.2\,dx}-\frac{\mathfrak{B}\,d^3X}{1.2.3.4\,dx^3}+\frac{\mathfrak{C}\,d^5X}{1.2.3.4.5.6\,dx^5}-\text{\&c.}$$

hinc facile erit iſtius functionis S differentiale exhibere erit enim:

$$dS=Xdx+\tfrac{1}{2}dX+\frac{\mathfrak{A}\,ddX}{1.2\,dx}-\frac{\mathfrak{B}\,d^4X}{1.2.3.4\,dx^3}+\frac{\mathfrak{C}\,d^6X}{1.2.3.4.5.6\,dx^5}-\text{\&c.}$$

387. Sin autem progreſſio propoſita coniuncta ſit cum geometrica, quo caſu termini eius infiniteſimi nunquam ad differentias conſtantes reducuntur, ac propterea methodus prior locum invenit nullum; tum methodus §. 174. tradita medelam afferet. Si enim propoſita ſit haec functio:

$S=Ap+Bp^2+Cp^3+Dp^4+\ldots+Xp^x$,

quaerantur valores litterarum α, β, γ, δ, &c. ut ſit

$$\frac{p-1}{p-e^u}=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\text{\&c.}$$

quibus inventis, uti eos §. 170. exhibuimus, erit:

$$S = \frac{p}{p-1} \cdot p^x \left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3 X}{dx^3} + \frac{\delta d^4 X}{dx^4} - \&c. \right)$$

± Conſtante, quae ſummam reddat $= 0$, ſi ponatur $x = 0$, ſeu quae cuiquam alii caſui ſatisfaciat. Sumto ergo differentiali haec conſtans ex computo abibit, eritque:

$$dS = \frac{p}{p-1} \cdot p^x dx lp \left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3 X}{dx^3} + \&c. \right)$$

$$+ \frac{p}{p-1} \cdot p^x \left(dX - \frac{\alpha ddX}{dx} + \frac{\beta d^3 X}{dx^2} - \frac{\gamma d^4 X}{dx^3} + \&c. \right) \quad \text{ſive}$$

$$dS = \frac{p^{x+1}}{p-1} \left(X dx lp - (\alpha lp - 1) dX + (\beta lp - \alpha) \frac{ddX}{dx} - (\gamma lp - \beta) \frac{d^3 X}{dx^2} + \&c. \right)$$

quod eſt differentiale quaeſitum functionis propoſitae S.

388. Sin autem functio inexplicabilis propoſita ex factoribus conſtet, eorumque logarithmi infiniteſimi differentias habeant conſtantes ſive minus; tum hac quoque methodo differentiale functionis perpetuo exhiberi poterit. Sit enim

$$S = \overset{1}{A}. \ \overset{2}{B}. \ \overset{3}{C}. \ \overset{4}{D}. \ \ldots \ \overset{x}{X}.$$

Quia hinc fit $lS = lA + lB + lC + lD + \ldots + lX$ methodo ſuperiori, numeros Bernoullianos in ſubſidium vocando erit:

$$lS = \int dx\, lX + \tfrac{1}{2} lX + \frac{\mathfrak{A}\, d.lX}{1.2\, dx} - \frac{\mathfrak{B}\, d^3.lX}{1.2.3.4\, dx^3} + \&c.$$

qua expreſſione differentiata fit:

$$\frac{dS}{S} = dx\, lX + \tfrac{1}{2} d.lX + \frac{\mathfrak{A}\, dd.lX}{1.2\, dx} - \frac{\mathfrak{B}\, d^4.lX}{1.2.3.4\, dx^3} +$$

$$\frac{\mathfrak{C}\, d^6.lX}{1.2.3.4.5.6\, dx^5} - \frac{\mathfrak{D}\, d^8.lX}{1.2.3 \ldots 8\, dx^7} + \&c.$$

Hinc ſi fuerit $X = x$, ut ſit:

$$S = 1.\ 2.\ 3.\ 4.\ \ldots\ x$$

fiet applicatione facta

$$\frac{dS}{S} = dx\, lx + \frac{dx}{2x} - \frac{\mathfrak{A}\, dx}{2xx} + \frac{\mathfrak{B}\, dx}{4x^4} - \frac{\mathfrak{C}\, dx}{6x^6} + \&c.$$

quae forma, ſi x ſit numerus valde magnus, commodius uſurpatur, quam eae, quas ante invenimus. CA-

CAPUT XVII.

DE INTERPOLATIONE SERIERUM.

389.

Series interpolari dicitur, dum eius termini aſſignantur, qui reſpondent indicibus fractis, vel etiam ſurdis. Si igitur ſeriei terminus generalis fuerit cognitus, interpolatio nullam habet difficultatem, cum quicunque numerus loco indicis x ſubſtituatur, iſta expreſſio praebeat terminum reſpondentem. Verum ſi ſeries ita fuerit comparata, ut eius terminus generalis nullo modo exhiberi queat; tum interpolatio huiuſmodi ſerierum plerumque eſt maxime difficilis, neque maximam partem termini indicibus non integris reſpondentes aliter niſi per ſeries infinitas definiri poſſunt. Quoniam ergo in Capite praecedente huiuſmodi expreſſionum, quae more conſueto finite exprimi non poſſunt, valores quibuſcunque indicibus reſpondentes determinavimus; ea tractatio maximam afferet utilitatem ad interpolationes perficiendas. Quam ob cauſam uſum, qui ex ſuperiori Capite in hoc negotium redundat, hic diligentius proſequemur.

390. Sit ergo propoſita ſeries quaecunque

$$\overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots . \overset{x}{X}$$

cuius terminus generalis X ſit cognitus, ſummatorius autem S lateat. Hinc formetur alia ſeries, cuius terminus generalis aequetur illius ſeriei termino ſummatorio, eritque iſta nova ſeries:

$$\overset{1}{A}; (\overset{2}{A+B}); (A+\overset{3}{B}+C); (A+B+\overset{4}{C}+D); (A+B+\overset{5}{C}+D+E); \text{ \&c.}$$

eiuſque terminus generalis ſeu indici indefinito x reſpondens

erit

erit $= A + B + C + D + \ldots + X = S$, qui cum explicite non ſit cognitus, interpolatio huius novae ſeriei iiſdem difficultatibus erit obnoxia, quas ante meminimus. Ad hanc ergo ſeriem interpolandam inveſtigari oportet valores ipſius S, quos recipit, ſi loco x numeri quicunque non integri ſubſtituantur. Si enim x eſſet numerus integer, tum conveniens ipſius S valor ſine difficultate reperiretur, additione ſcilicet tot terminorum ſeriei $A + B + C + D +$ &c. quot x contineat unitates.

391. Quo igitur ea, quae in Capite praecedente ſunt tradita, in uſum vocari poſſint, ponamus x eſſe numerum integrum, ita ut valor ei reſpondens $S = A + B + C + \ldots + X$ ſit cognitus, & quaeramus valorem Σ, in quem S tranſmutetur, ſi loco x ſcribatur $x + \omega$, exiſtente ω fractione quacunque; eritque Σ terminus ſeriei propoſitae interpolandae, qui reſpondet indici $x + \omega$; quo ergo invento, interpolatio huius ſeriei erit in promtu. Sit Z terminus ſeriei A, B, C, D, E, &c. qui reſpondet indici $x + \omega$, ſintque Z', Z'', Z''', &c. termini eius conſecutivi indices habentes $x + \omega + 1$; $x + \omega + 2$: $x + \omega + 3$; &c. Ac primo quidem ponamus ſeriei A, B, C, D, &c. terminos infiniteſimos evaneſcere. His ergo poſitis ſeries

$$\overset{1}{A};\ (A\overset{2}{+}B);\ (A+\overset{3}{B}+C);\ (A+B\overset{4}{+}C+D);\ \text{\&c.}$$

cuius terminus indici x reſpondens eſt

$$S = A + B + C + \ldots + X$$

interpolabitur quaerendo eius terminum Σ, qui indici fracto $x + \omega$ reſpondeat, erit autem uti invenimus:

$$\Sigma = S \begin{matrix} + X' + X'' + X''' + X'''' + \text{\&c.} \\ - Z' - Z'' - Z''' - Z'''' - \text{\&c.} \end{matrix}$$

ſicque habebitur ſeries infinita iſti termino quaeſito Σ aequalis, quae ob

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1.2\,dx^2} + \frac{\omega^3 d^3X}{1.2.3\,dx^3} + \text{\&c.}$$

in

in hanc formam tranſmutatur, ut ſit:

$$\Sigma = S - \frac{\omega}{dx} d. (X' + X'' + X''' + X'''' + \&c.)$$
$$- \frac{\omega^2}{2dx^2} dd. (X' + X'' + X''' + X'''' + \&c.)$$
$$- \frac{\omega^3}{6dx^3} d^3. (X' + X'' + X''' + X'''' + \&c.)$$
&c.

quarum formularum ea, quae quovis caſu commodior videatur, adhiberi poterit.

392. Sumamus pro A, B, C, D &c. ſeriem harmonicam quamcunque $\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} +$ &c. cuius terminus generalis ſeu indici x reſpondens eſt $= \frac{1}{a+(x-1)b} = X$. Hinc formata ſit iſta ſeries:

$$\overset{1}{\frac{1}{a}}; \overset{2}{\left(\frac{1}{a}+\frac{1}{a+b}\right)}; \overset{3}{\left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}\right)}; \overset{4}{\left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}\right)} \&c.$$

cuius propterea terminus indici x reſpondens erit:

$$S = \frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+(x-1)b}.$$

Si iam Σ denotet terminum iſtius ſeriei indici $x + \omega$ reſpondentem, ob $Z = \frac{1}{a+(x+\omega-1)b}$, erit

$$X' = \frac{1}{a+bx};\quad Z' = \frac{1}{a+bx+b\omega}$$
$$X'' = \frac{1}{a+b+bx};\quad Z'' = \frac{1}{a+b+bx+b\omega}$$
$$X''' = \frac{1}{a+2b+bx};\quad Z''' = \frac{1}{a+2b+bx+b\omega}$$
&c. &c.

hinc-

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hincque orietur:

$$\Sigma = S + \frac{1}{a+bx} + \frac{1}{a+b+bx} + \frac{1}{a+2b+bx} + \&c.$$
$$- \frac{1}{a+bx+b\omega} - \frac{1}{a+b+bx+b\omega} - \frac{1}{a+2b+bx+b\omega} - \&c.$$

altera expreſſio autem erit huiuſmodi:

$$\Sigma = +b\,\omega\left(\frac{1}{(a+bx)^2} + \frac{1}{(a+b+bx)^2} + \frac{1}{(a+2b+bx)^2} + \&c.\right)$$
$$-b^2\omega^2\left(\frac{1}{(a+bx)^3} + \frac{1}{(a+b+bx)^3} + \frac{1}{(a+2b+bx)^3} + \&c.\right)$$
$$+b^3\omega^3\left(\frac{1}{(a+bx)^4} + \frac{1}{(a+b+bx)^4} + \frac{1}{(a+2b+bx)^4} + \&c.\right)$$
$$\&c.$$

EXEMPLUM I.

Propoſita ſit iſta ſeries

$$\overset{1}{1};\ \overset{2}{(1+\tfrac{1}{2})};\ \overset{3}{(1+\tfrac{1}{2}+\tfrac{1}{3})};\ \overset{4}{(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})};\ \&c.$$

cuius terminos, qui indicibus fractis reſpondent, inveniri oporteat.

Erit ergo $a = 1$ & $b = 1$; unde ſi terminus indici integro x reſpondens ponatur

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \quad \ldots \quad + \frac{1}{x};$$

terminuſque indici fracto $x + \omega$ reſpondens vocetur $= \Sigma$, erit:

$$\Sigma = S + \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \frac{1}{4+x} + \frac{1}{5+x} + \&c.$$
$$- \frac{1}{1+x+\omega} - \frac{1}{2+x+\omega} - \frac{1}{3+x+\omega} - \frac{1}{4+x+\omega} - \frac{1}{5+x+\omega} - \&c.$$

Notandum autem eſt, ſi inventus fuerit terminus reſpondens indici fracto ω, quem ponamus $= T$, ex eo terminum indicis

dicis $x+\omega$ facile inveniri poſſe; erit enim ſi T', T'', T''', &c. denotent terminos indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. reſpondentes:

$$T' = T + \frac{1}{1+\omega}$$

$$T'' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega}$$

$$T''' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega} + \frac{1}{3+\omega} \quad ; \qquad \&c.$$

unde ſufficit eos tantum terminos, qui reſpondent indicibus ω unitate minoribus, inveſtigaſſe. Quem in finem ponamus $x=0$, erit quoque $S=0$, atque terminus ſeriei T indici fracto ω reſpondens ita exprimetur:

$$T = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$$
$$- \frac{1}{1+\omega} - \frac{1}{2+\omega} - \frac{1}{3+\omega} - \frac{1}{4+\omega} - \&c.$$

vel his fractionibus in ſeries infinitas converſis prodibit altera expreſſio:

$$T = +\omega \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$
$$-\omega^2 \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+\omega^3 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$
$$-\omega^4 \left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \&c.\right)$$
$$\&c.$$

quae ad valorem ipſius T proxime inveniendum perquam eſt apta.

Quaeratur ergo propoſitae ſeriei terminus reſpondens indici $\frac{1}{2}$, qui ſi ponatur $= T$, erit: $T =$

$$T = 1 - \tfrac{2}{3} + \tfrac{1}{2} - \tfrac{2}{5} + \tfrac{1}{3} - \tfrac{2}{7} + \tfrac{1}{4} - \tfrac{2}{9} + \&c.$$

ſeu $T = 2\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{6} - \&c.\right)$

cuius ſeriei valor eſt $= 2 - 2l2$, ſicque terminus indicis $= \tfrac{1}{2}$ finite exprimi poteſt. Erunt ergo termini ſequentes, quorum indices ſunt $\tfrac{1}{2}$, $\tfrac{3}{2}$, $\tfrac{5}{2}$, $\tfrac{7}{2}$, &c. ita expreſſi:

Ind.	$\tfrac{1}{2}$	$\tfrac{3}{2}$	$\tfrac{5}{2}$	$\tfrac{7}{2}$	
Term.	$2 - 2l2$;	$2 + \tfrac{2}{3} - 2l2$;	$2 + \tfrac{2}{3} + \tfrac{2}{5} - 2l2$;	$2 + \tfrac{2}{3} + \tfrac{2}{5} + \tfrac{2}{7} - 2l2$;	&c.

EXEMPLUM II.

Propoſita ſit iſta ſeries:

$$\overset{1}{1};\ (1 \overset{2}{+} \tfrac{1}{3});\ (1 \overset{3}{+} \tfrac{1}{3} + \tfrac{1}{5});\ (1 + \tfrac{1}{3} \overset{4}{+} \tfrac{1}{5} + \tfrac{1}{7});\ \&c.$$

cuius terminos indicibus fractis reſpondentes exprimere oporteat:

Erit ergo $a = 1$, $b = 2$, unde ſi terminus indici integro x reſpondens ponatur

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \quad . \quad . \quad . \quad + \frac{1}{2x - 1}$$

terminuſque indici fracto $x + \omega$ vocetur $= \Sigma$, erit

$$\Sigma = S + \frac{1}{1 + 2x} + \frac{1}{3 + 2x} + \frac{1}{5 + 2x} + \frac{1}{7 + 2x} + \&c.$$
$$- \frac{1}{1 + 2(x + \omega)} - \frac{1}{3 + 2(x + \omega)} - \frac{1}{5 + 2(x + \omega)} - \frac{1}{7 + 2(x + \omega)} - \&c.$$

Cum igitur ſufficiat terminos indicibus unitate minoribus aſſignaſſe, ſit $x = 0$, & $S = 0$: quocirca ſi terminus indici ω conveniens ponatur $= T$, erit:

$$T = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c.$$
$$- \frac{1}{1 + 2\omega} - \frac{1}{3 + 2\omega} - \frac{1}{5 + 2\omega} - \frac{1}{7 + 2\omega} - \frac{1}{9 + 2\omega} - \&c.$$

& ſi ω numerum quemcunque denotare ponatur, quoniam T eſt terminus indici ω reſpondens, erit T terminus generalis ſeriei propoſitae, qui etiam hoc modo exprimetur:

$$T = \frac{2\omega}{1(1 + 2\omega)} + \frac{2\omega}{3(3 + 2\omega)} + \frac{2\omega}{5(5 + 2\omega)} + \frac{2\omega}{7(7 + 2\omega)} + \&c. \text{ vel ita}$$

$T =$

$$T = 2\omega\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.\right)$$
$$-4\omega^2\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \&c.\right)$$
$$+8\omega^3\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.\right)$$
$$-16\omega^4\left(1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} + \&c.\right)$$
$$\&c.$$

Ponamus eſſe $\omega = \frac{1}{2}$, erit terminus huic indici reſpondens

$$T = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \&c. = l\,2\text{, eruntque}$$

Ind.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Term.	$l\,2$;	$\frac{1}{2} + l\,2$;	$\frac{1}{2} + \frac{1}{4} + l\,2$;	$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + l\,2$;	&c.

Si ſit $\omega = \frac{1}{4}$; erit

$$T = \begin{array}{l} +\,1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c. \\ -\,\frac{2}{3} - \frac{2}{7} - \frac{2}{11} - \frac{2}{15} - \&c. \end{array}$$

ſive $T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. = \frac{\pi}{4}$

393. Quod ſi ergo huius ſeriei generalis:

$$\frac{1}{a}\,;\ \left(\frac{1}{a} + \frac{1}{a+b}\right);\ \left(\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b}\right);\ \&c.$$

quaeratur terminus reſpondens indici $= \frac{1}{2}$, ponatur in expreſſionibus §. praeced. $x = 0$, & $\omega = \frac{1}{2}$; fietque $S = 0$, & terminus indici $\frac{1}{2}$ reſpondens quaeſitus erit

$$\Sigma = \frac{1}{a} - \frac{2}{2a+b} + \frac{1}{a+b} - \frac{2}{2a+3b} + \frac{1}{a+2b} - \frac{2}{2a+5b} + \&c.$$

ſive terminis ad maiorem uniformitatem perductis erit

$$\tfrac{1}{2}\Sigma = \frac{1}{2a} - \frac{1}{2a+b} + \frac{1}{2a+2b} - \frac{1}{2a+3b} + \frac{1}{2a+4b} - \&c.$$

in qua ſerie cum ſigna + & — alternentur, ſumendis continuis differentiis per methodum ſupra expoſitam valor ipſius $\frac{1}{2}\Sigma$ per ſeriem magis convergentem exprimetur. Erunt autem differentiarum ſeries: **b**

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$$\frac{b}{2a(2a+b)};\ \frac{b}{(2a+b)(2a+2b)};\ \frac{b}{(2a+2b)(2a+3b)};\ \&c.$$

$$\frac{2bb}{2a(2a+b)(2a+2b)};\ \frac{2bb}{(2a+b)(2a+2b)(2a+3b)};\ \&c.$$

$$\frac{6b^3}{2a(2a+b)(2a+2b)(2a+3b)};\ \&c.$$

&c.

Ex quibus concluditur fore:

$$\tfrac{1}{2}\Sigma = \frac{1}{4a} + \frac{1b}{8a(2a+b)} + \frac{1.\ 2bb}{16a(2a+b)(2a+2b)} + \frac{1.\ 2.\ 3b^3}{32a(2a+b)(2a+2b)(2a+3b)} + \&c.$$

Hincque ergo habebitur:

$$\Sigma = \frac{1}{2a} + \frac{\frac{1}{2}.b}{2a(2a+b)} + \frac{\frac{1}{2}.\frac{2}{2}bb}{2a(2a+b)(2a+2b)} + \frac{\frac{1}{2}.\frac{2}{2}.\frac{3}{2}b^3}{2a(2a+b)(2a+2b)(2a+3b)} + \&c.$$

quae ſeries maxime convergit, atque valorem termini Σ facili labore proxime exhibet.

394. Quod ſi autem in genere ſeriei A, B, C, D, E, &c. termini infiniteſimi evaneſcant, terminuſque indici ω reſpondens fuerit $=Z$, eiuſque ſequentes, qui indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. reſpondeant, ſint Z', Z'', Z''', Z^{IV}, &c. Si in ſuperioribus (391) ponatur $x=0$, ut ſit $S=0$ & $X'=A$, $X''=B$, $X'''=C$, &c. ſequetur, ſi formetur huiuſmodi ſeries:

$$\overset{1}{A},\ (A\overset{2}{+}B),\ (A+\overset{3}{B}+C),\ (A+B\overset{4}{+}C+D),\ \&c.$$

eiuſque terminus indici ω reſpondens ponatur $=\Sigma$, fore

$$\Sigma = (A-Z') + (B-Z'') + (C-Z''') + (D-Z^{IV}) + \&c.$$

ex qua expreſſione termini quicunque intermedii definiri poterunt. Sufficiet autem ad interpolationem perficiendam eos terminos inveſtigaſſe, qui reſpondeant indicibus ω unitate minori-

noribus. Si enim terminus Σ indici huiuſmodi cuicunque ω reſpondens fuerit repertus, iique qui conveniant indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. ponantur Σ', Σ'', Σ''', Σ^{IV} &c. erit

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + Z''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z'''$$
&c.

EXEMPLUM I.

Interpolare hanc ſeriem:

$$\overset{1}{1};\ \overset{2}{(1+\tfrac{1}{4})};\ \overset{3}{(1+\tfrac{1}{4}+\tfrac{1}{9})};\ \overset{4}{(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16})};$$
&c.

Sit Σ huius ſeriei terminus reſpondens indici ω, & cum haec ſeries formata ſit ex ſummatione huius:

$$1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\text{\&c.}$$

cuius terminus indici ω reſpondens eſt $=\frac{1}{\omega^2}$ erit

$$\Sigma = +\ 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{\&c.}$$
$$-\frac{1}{(1+\omega)^2}-\frac{1}{(2+\omega)^2}-\frac{1}{(3+\omega)^2}-\frac{1}{(4+\omega)^2}-\text{\&c.}$$

Quod ſi ergo ſeriei propoſitae quaeratur terminus indici $\frac{1}{2}$ reſpondens, poni debebit $\omega=\frac{1}{2}$, fietque:

$\Sigma = 1-\frac{4}{9}+\frac{1}{4}-\frac{4}{25}+\frac{1}{9}-\frac{4}{49}+\text{\&c.}$ ſive

$\Sigma = 4(\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\frac{1}{25}+\frac{1}{36}-\frac{1}{49}+\text{\&c.})$

Cum igitur ſit $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{\&c.} = \frac{\pi^2}{12}$; erit

$\Sigma = 4\left(1-\frac{\pi\pi}{12}\right) = 4-\frac{1}{3}\pi^2$, qui eſt terminus indici $\frac{1}{2}$ reſpondens. Hinc ergo reſpondebunt

Indicibus	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	&c.
Termini	$4-\frac{1}{3}\pi^2$;	$\frac{4}{1}+\frac{4}{9}-\frac{1}{3}\pi^2$;	$\frac{4}{1}+\frac{4}{9}+\frac{4}{25}-\frac{1}{3}\pi^2$;	&c.

EXEMPLUM II.

Interpolare hanc ſeriem:

$$\overset{1}{1};\ \overset{2}{(1+\tfrac{1}{9})};\ \overset{3}{(1+\tfrac{1}{9}+\tfrac{1}{25})};\ \overset{4}{(1+\tfrac{1}{9}+\tfrac{1}{25}+\tfrac{1}{49})}\ \ \&c.$$

Sit Σ terminus reſpondens indici cuicunque ω, & cum haec ſeries formata ſit ex ſummatione huius:

$$1+\tfrac{1}{9}+\tfrac{1}{25}+\tfrac{1}{49}+\tfrac{1}{81}+\&c.$$

ex qua ſit terminus indici ω reſpondens $Z=\frac{1}{(2\omega-1)^2}$

erit $Z'=\frac{1}{(2\omega+1)^2};\ Z''=\frac{1}{(2\omega+3)^2};\ Z'''=\frac{1}{(2\omega+5)^2}$ &c.

Quamobrem habebitur:

$$\Sigma = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \&c.$$
$$-\frac{1}{(1+2\omega)^2}-\frac{1}{(3+2\omega)^2}-\frac{1}{(5+2\omega)^2}-\frac{1}{(7+2\omega)^2}-\&c.$$

Ponamus $\omega=\frac{1}{2}$, ut inveniamus terminum ſeriei propoſitae reſpondentem indici $=\frac{1}{2}$, qui erit:

$$\Sigma=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\&c.=\frac{\pi\pi}{12},$$

ex quo termini, qui medium interiacent inter binos quoſvis datos, ſequenti modo exprimentur. Reſpondebunt

Ind. $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; $\frac{7}{2}$; &c.

Term. $\frac{\pi\pi}{12};\ \frac{1}{4}+\frac{\pi\pi}{12};\ \frac{1}{4}+\frac{1}{16}+\frac{\pi\pi}{12};\ \frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{\pi\pi}{12}$; &c.

EXEMPLUM III.

Interpolare hanc ſeriem:

$$\overset{1}{1};\ \overset{2}{\left(1+\frac{1}{2^n}\right)};\ \overset{3}{\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)};\ \overset{4}{\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)};\ \&c.$$

Sit ut ante Σ terminus indici ω reſpondens, erit

$Z=$

$$Z = \frac{1}{\omega^n};\ \&\ Z' = \frac{1}{(1+\omega)^n};\ Z'' = \frac{1}{(2+\omega)^n};\ Z''' = \frac{1}{(3+\omega)^n};\ \&c.$$

hincque habebitur:

$$\Sigma = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$$
$$- \frac{1}{(1+\omega)^n} - \frac{1}{(2+\omega)^n} - \frac{1}{(3+\omega)^n} - \frac{1}{(4+\omega)^n} - \&c.$$

Si igitur deſideretur terminus indici $\frac{1}{2}$ reſpondens: erit is

$$= 1 - \frac{2^n}{3^n} + \frac{1}{2^n} - \frac{2^n}{5^n} + \frac{1}{3^n} - \frac{2^n}{7^n} + \&c.$$

$$\text{ſeu} = 2^n \left(\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \&c. \right)$$

Quare ſi ponatur:

$$\mathfrak{N} = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \&c.$$

erit ſeriei propoſitae terminus qui indici $\frac{1}{2}$ reſpondet $= 2^n(1 - \mathfrak{N})$; hincque reſpondebunt

Indic. $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; &c.

Term. $2^n - 2^n\mathfrak{N}$; $2^n + \frac{2^n}{3^n} - 2^n\mathfrak{N}$; $2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} - 2^n\mathfrak{N}$; &c.

EXEMPLUM IV.

Interpolare hanc ſeriem:

1 2 3 4

$$1;\ \left(1 + \frac{1}{3^n}\right);\ \left(1 + \frac{1}{3^n} + \frac{1}{5^n}\right);\ \left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n}\right);\ \&c.$$

Sit Σ terminus qui indici cuicunque ω reſpondeat, & cum ſit $Z = \frac{1}{(2\omega - 1)^n}$, erit:

$$Z' = \frac{1}{(2\omega + 1)^n};\ Z'' = \frac{1}{(2\omega + 3)^n};\ Z''' = \frac{1}{(2\omega + 5)^n};\ \&c.$$

 at-

atque $\Sigma = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \&c.$

$$- \frac{1}{(1+2\omega)^n} - \frac{1}{(3+2\omega)^n} - \frac{1}{(5+2\omega)^n} - \frac{1}{(7+2\omega)^n} - \&c.$$

Ponatur $\omega = \frac{1}{2}$, & prodibit terminus indici $\frac{1}{2}$ reſpondens

$$= 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \&c. = \mathfrak{N}$$

ex quo porro erunt reliqui termini inter binos datos medii

Indices:	$\frac{1}{2}$;	$\frac{3}{2}$;	$\frac{5}{2}$;	&c.
Termini:	$\mathfrak{N}$;	$\frac{1}{2^n} + \mathfrak{N}$;	$\frac{1}{2^n} + \frac{1}{4^n} + \mathfrak{N}$;	&c.

395. Ponamus nunc ſeriei A, B, C, D, E, &c. ex cuius ſummatione ſeries interpolanda formatur, terminos infiniteſimos non evaneſcere, ſed ita eſſe comparatos, ut eorum differentiae evaneſcant; ſitque X huius ſeriei terminus reſpondens indici x, & Z terminus reſpondens exponenti $x+\omega$, tum vero ſint X', X'', X''', X'''', &c. termini ipſum X ſequentes, & Z', Z'', Z''', &c. termini ipſum Z ſequentes. Quibus poſitis proponatur haec ſeries interpolanda:

$$\overset{1}{A};\ \overset{2}{(A+B)};\ \overset{3}{(A+B+C)};\ \overset{4}{(A+B+C+D)};\ \&c.$$

cuius terminus indici x reſpondens ſit $= S$, at terminus indici $x+\omega$ reſpondens ſit $= \Sigma$; eritque ex iis, quae Capite praecedente ſunt tradita:

$$\begin{aligned}\Sigma = {} & S + X' + X'' + X''' + \&c. \\ & - Z' - Z'' - Z''' - \&c. \\ & + \omega X' + \omega \left\{ \begin{matrix} X'' + X''' + X'''' + \&c. \\ -X' - X'' - X''' - \&c. \end{matrix} \right\}\end{aligned}$$

Quia autem ut ante ſufficit terminos indicibus unitate minoribus reſpondentes inveſtigaſſe, ponamus $x = 0$, ut ſit $S = 0$, $X' = A$, $X'' = B$, &c. eritque terminus indici ω reſpondens:

$$\begin{aligned}\Sigma = {} & (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''')\ \&c. \\ & + \omega A + \omega[(B-A) + (C-B) + (D-C) + (E-D) + \&c.]\end{aligned}$$

Vel ſi differentias has more ſupra recepto exprimere velimus quo eſt $\Delta A = B - A$; $\Delta B = C - B$; &c. habebitur:

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') + \&c.$$
$$+ \omega (A + \Delta A + \Delta B + \Delta C + \Delta D + \&c.)$$

396. Sin autem ſeriei A, B, C, D, E, &c. ex cuius ſummatione ſeries interpolanda formatur, termini infinitesimi neque ipſi evaneſcant, neque differentias primas habeant evaneſcentes; tum plures ſeries ad valorem ipſius Σ exprimendum adiici debebunt, quoad ſcilicet ad differentias terminorum infiniteſimorum evaneſcentes perveniatur. Sit enim ut ante ſeriei A, B, C, D, E, &c. terminus indici x reſpondens $= X$, eumque ſequentes X', X'', X''', &c. indici autem $x + \omega$ reſpondeat terminus Z, quem ſequantur Z', Z'', &c. atque proponatur haec ſeries:

$$\overset{1}{A}\,;\ \overset{2}{(A + B)}\,;\ \overset{3}{(A + B + C)}\,;\ \overset{4}{(A + B + C + D)}\,;\ \&c.$$

cuius terminus indici x reſpondens ſit

$$S = A + B + C + D + \ldots\ldots + X$$

indici vero $x + \omega$ reſpondeat terminus Σ; ita ut

indicibus	reſpondeant termini
$x + \omega + 1$	$\Sigma' = \Sigma + Z'$
$x + \omega + 2$	$\Sigma'' = \Sigma + Z' + Z''$
$x + \omega + 3$	$\Sigma''' = \Sigma + Z' + Z'' + Z'''$
&c.	&c.

Si iam differentiae terminorum ita exprimantur, ut ſit

$$\Delta X' = X'' - X';\ \Delta X'' = X''' - X'';\ \Delta X''' = X'''' - X''';\ \&c.$$
$$\Delta^2 X' = \Delta X'' - \Delta X';\ \Delta^2 X'' = \Delta X''' - \Delta X'';\ \Delta^2 X''' = \Delta X'''' - \Delta X''',\ \&c.$$
$$\Delta^3 X' = \Delta^2 X'' - \Delta^2 X';\ \Delta^3 X'' = \Delta^2 X''' - \Delta^2 X'';\ \&c.$$

ex §. 377. terminus Σ ſequenti modo exprimetur:

$$\Sigma = S \quad + X' + X'' + X''' + X'''' + \&c.$$
$$\qquad\qquad - Z' - Z'' - Z''' - Z'''' - \&c.$$
$$+ \omega [X' + \Delta X' + \Delta X'' + \Delta X''' + \Delta X'''' + \&c.]$$
$$+ \frac{\omega(\omega - 1)}{1 \cdot 2} [\Delta X' + \Delta^2 X' + \Delta^2 X'' + \Delta^2 X''' + \Delta^2 X'''' + \&c.]$$
$$+ \frac{\omega(\omega - 1)(\omega - 2)}{1 \cdot 2 \cdot 3} [\Delta^2 X' + \Delta^3 X' + \Delta^3 X'' + \Delta^3 X''' + \Delta^3 X'''' + \&c.]$$
$$\&c.$$

397. Sufficit, uti iam notavimus, tot huiuſmodi ſeries adieciſſe, donec ad terminorum infiniteſimorum differentias evaneſcentes perveniatur: ſi enim has ipſas ſeries quoque in infinitum continuare velimus, vel eo uſque ſaltem, donec terminorum finitorum differentiae evaneſcant; tum ob

$$Z' = X' + \omega\Delta X' + \frac{\omega(\omega-1)}{1.\ 2}\Delta^2 X' + \frac{\omega(\omega-1)(\omega-2)}{1.\ 2.\ 3}\Delta^3 X' + \&c.$$

tota expreſſio inventa contrahetur in hanc:

$$\Sigma = S + \omega X' + \frac{\omega(\omega-1)}{1.\ 2}\Delta X' + \frac{\omega(\omega-1)(\omega-2)}{1.\ 2.\ 3}\Delta^2 X' + \&c.$$

quae terminum ſummatorium ſeriei $A + B + C + D +$ &c. involvit; qui autem ſi eſſet cognitus, interpolatio nullam haberet difficultatem. Interim tamen & hac formula uti licebit, quippe quae, quoties abrumpitur, quemvis terminum interpolandum finite & algebraice expreſſum exhibet: ſin autem in infinitum progrediatur, plerumque praeſtat priorem formulam adhibere, in qua ratio terminorum infiniteſimorum habetur. Haec vero, ſi ponatur $x = 0$, ut Σ denotet terminum indici ω reſpondentem, ob $S = 0$ hanc formam induet:

$$\begin{aligned} \Sigma = \quad & + A + B + C + D + \&c. \\ & - Z' - Z'' - Z''' - Z'''' - \&c. \\ & + \omega\,[A + \Delta A + \Delta B + \Delta C + \Delta D + \&c.] \\ & + \frac{\omega(\omega-1)}{1.\ 2}[\Delta A + \Delta^2 A + \Delta^2 B + \Delta^2 C + \Delta^2 D + \&c.] \\ & + \frac{\omega(\omega-1)(\omega-2)}{1.\ 2.\ 3}[\Delta^2 A + \Delta^3 A + \Delta^3 B + \Delta^3 C + \Delta^3 D + \&c.] \\ & \qquad \&c. \end{aligned}$$

Vel ſi ponatur brevitatis gratia:

$$\omega = \alpha; \quad \frac{\omega(\omega-1)}{1.\ 2} = \beta; \quad \frac{\omega(\omega-1)(\omega-2)}{1.\ 2.\ 3} = \gamma; \ \&c.$$

erit

$$\begin{aligned} \Sigma = \quad & \alpha A + \beta\Delta A + \gamma\Delta^2 A + \delta\Delta^3 A + \&c. \\ & + A + \alpha\Delta A + \beta\Delta^2 A + \gamma\Delta^3 A + \&c. - Z' \\ & + B + \alpha\Delta B + \beta\Delta^2 B + \gamma\Delta^3 B + \&c. - Z'' \\ & + C + \alpha\Delta C + \beta\Delta^2 C + \gamma\Delta^3 C + \&c. - Z''' \\ & \qquad \&c. \end{aligned}$$

qua-

quarum ſerierum horizontalium numerus in infinitum quidem progreditur, at quaelibet finito terminorum numero conſtat.

EXEMPLUM.

Interpolare hanc ſeriem:

$$\overset{1}{\tfrac{1}{2}};\ \overset{2}{\tfrac{1}{2}+\tfrac{2}{3}};\ \overset{3}{\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}};\ \overset{4}{\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}+\tfrac{4}{5}};\ \&c.$$

Sit huius ſeriei terminus indici ω reſpondens $= \Sigma$, & cum ea oriatur ex ſummatione huius ſeriei:

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \&c. \quad \text{erit} \quad Z = \frac{\omega}{\omega+1};$$

& quia termini infiniteſimi differentias ſuas primas iam habent evaneſcentes, differentiae tantum primae ſunt accipiendae, quae erunt:

$$\text{ob } A = \frac{1}{2};\ B = \frac{2}{3};\ C = \frac{3}{4};\ D = \frac{4}{5};\ \&c.$$

$$\Delta A = \frac{1}{2\cdot 3};\ \Delta B = \frac{1}{3\cdot 4};\ \Delta C = \frac{1}{4\cdot 5};\ \&c.$$

Hinc ergo habebitur:

$$\begin{aligned}\Sigma = \frac{\omega}{2} &+ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \&c.\\ &+ \frac{\omega}{2\cdot 3} + \frac{\omega}{3\cdot 4} + \frac{\omega}{4\cdot 5} + \frac{\omega}{5\cdot 6} + \&c.\\ &- \frac{\omega+1}{\omega+2} - \frac{\omega+2}{\omega+3} - \frac{\omega+3}{\omega+4} - \frac{\omega+4}{\omega+5} - \&c.\end{aligned}$$

ſeu ob $\frac{\omega}{2} + \frac{\omega}{2\cdot 3} + \frac{\omega}{3\cdot 4} + \frac{\omega}{4\cdot 5} + \&c. = \omega$; erit

$$\begin{aligned}\Sigma = \omega &+ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \&c.\\ &- \frac{\omega+1}{\omega+2} - \frac{\omega+2}{\omega+3} - \frac{\omega+3}{\omega+4} - \frac{\omega+4}{\omega+5} - \&c.\end{aligned}$$

Si ergo quaeratur terminus indici $\frac{1}{2}$ reſpondens, erit is

$\Sigma =$

$$\Sigma = \frac{1}{2} + \frac{1}{2} - \frac{3}{5} + \frac{2}{3} - \frac{5}{7} + \frac{3}{4} - \frac{7}{9} + \frac{4}{5} - \frac{9}{11} + \&c.$$

$$\text{ſeu } \Sigma = \frac{1}{2} - \frac{1}{2.5} - \frac{1}{3.7} - \frac{1}{4.9} - \frac{1}{5.11} - \frac{1}{6.13} - \&c.$$

$$\text{ideoque } \tfrac{1}{2}\Sigma = \frac{1}{4} - \frac{1}{4.5} - \frac{1}{6.7} - \frac{1}{8.9} - \frac{1}{10.11} - \frac{1}{12.13} - \&c.$$

$$\text{ſeu } \tfrac{1}{2}\Sigma = \frac{1}{4} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \&c.$$
$$+ \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \&c.$$

Quare cum ſit $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c. = l\,2$

erit $\frac{1}{2}\Sigma = l\,2 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = l\,2 - \frac{7}{12}$,

ideoque $\Sigma = 2\,l\,2 - \frac{7}{6}$.

398. Pergamus nunc ad ſeries interpolandas, quarum termini ex factoribus ſunt conflati, ſitque propoſita haec ſeries generaliſſima:

$$\overset{1}{A};\ \overset{2}{AB};\ \overset{3}{ABC};\ \overset{4}{ABCD};\ \overset{5}{ABCDE};\ \&c.$$

cuius terminus indici ω reſpondens ſit $= \Sigma$. Erit ergo $l\,\Sigma$ terminus reſpondens indici ω in hac ſerie:

$$\overset{1}{lA};\ \overset{2}{(lA + lB)};\ \overset{3}{(lA + lB + lC)};\ \overset{4}{(lA + lB + lC + lD)};\ \&c.$$

Quodſi ergo ponamus huius ſeriei terminos infiniteſimos evaneſcere; atque ſeriei A, B, C, D, E, &c. terminum indici ω reſpondentem eſſe Z, eiuſque ſequentes indicibus $\omega+1, \omega+2, \omega+3$, &c. reſpondentes eſſe Z', Z'', Z''', Z'''', &c. erit ex ſupra demonſtratis:

$$l\,\Sigma = \begin{matrix} + lA + lB + lC + lD + \&c. \\ - lZ' - lZ'' - lZ''' - lZ'''' - \&c. \end{matrix}$$

Hinc

Hinc igitur ad numeros progrediendo habebitur:

$$\Sigma = \frac{A}{Z'} \cdot \frac{B}{Z''} \cdot \frac{C}{Z'''} \cdot \frac{D}{Z''''}, \text{ \&c.}$$

399. Quodſi autem terminorum infiniteſimorum ſeriei A, B, C, D, &c. logarithmi non evaneſcant, ſed habeant differentias evaneſcentes, erit uti vidimus:

$$\begin{aligned} l\Sigma = &+ lA + lB + lC + \text{\&c.} \\ &- lZ' - lZ'' - lZ''' - \text{\&c.} \\ &+ \omega lA + \omega \left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + \text{\&c.} \right) \end{aligned}$$

hincque ad numeros a logarithmis procedendo fiet

$$\Sigma = A^{\omega} \cdot \frac{A^{1-\omega} B^{\omega}}{Z'} \cdot \frac{B^{1-\omega} C^{\omega}}{Z''} \cdot \frac{C^{1-\omega} D^{\omega}}{Z'''} \cdot \frac{D^{1-\omega} E^{\omega}}{Z''''} \cdot \text{\&c.}$$

At ſi illorum logarithmorum infiniteſimorum differentiae demum ſecundae evaneſcant, erit:

$$\begin{aligned} l\Sigma = \quad & lA + lB + lC + lD + \text{\&c.} \\ & - lZ' - lZ'' - lZ''' - lZ'''' - \text{\&c.} \\ & + \omega \left(lA + l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \text{\&c.} \right) \\ & + \frac{\omega(\omega - 1)}{1 \cdot 2} \left(l\frac{B}{A} + l\frac{AC}{B^2} + l\frac{BD}{C^2} + l\frac{CE}{D^2} + l\frac{DE}{E^2} + \text{\&c.} \right) \end{aligned}$$

Ex his itaque obtinebitur:

$$\Sigma = A^{\frac{\omega(3-\omega)}{2}} \cdot B^{\frac{\omega(\omega-1)}{1 \cdot 2}} \cdot \frac{A^{\frac{(\omega-1)(\omega-2)}{1 \cdot 2}} B^{\omega(2-\omega)} C^{\frac{\omega(\omega-1)}{1 \cdot 2}}}{Z'} \cdot \frac{B^{\frac{(\omega-1)(\omega-2)}{1 \cdot 2}} C^{\omega(2-\omega)} D^{\frac{\omega(\omega-1)}{1 \cdot 2}}}{Z''} \text{ \&c.}$$

quae ſi $\omega < 1$ commodius ita exprimetur:

$$\Sigma = \frac{A^{\frac{\omega(3-\omega)}{1 \cdot 2}}}{B^{\frac{\omega(1-\omega)}{1 \cdot 2}}} \cdot \frac{A^{\frac{(1-\omega)(2-\omega)}{1 \cdot 2}} B^{\omega(2-\omega)}}{C^{\frac{\omega(1-\omega)}{1 \cdot 2}} Z'} \cdot \frac{B^{\frac{(1-\omega)(2-\omega)}{1 \cdot 2}} C^{\omega(2-\omega)}}{D^{\frac{\omega(1-\omega)}{1 \cdot 2}} Z''} \cdot \text{\&c.}$$

400. Accommodemus hanc interpolationem ad iſtam ſeriem:

$$\overset{1}{\frac{a}{b}};\ \overset{2}{\frac{a(a+c)}{b(b+c)}};\ \overset{3}{\frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}};\ \overset{4}{\frac{a(a+c)(a+2c)(a+3c)}{b(b+c)(b+2c)(b+3c)}};\ \&c.$$

cuius factores deſumti ſunt ex hac ſerie:

$$\overset{1}{\frac{a}{b}};\ \overset{2}{\frac{a+c}{b+c}};\ \overset{3}{\frac{a+2c}{b+2c}};\ \overset{4}{\frac{a+3c}{b+3c}};\ \&c.$$

cuius terminorum infiniteſimorum logarithmi ſunt $=0$.

Erit ergo $Z=\frac{a-c+c\omega}{b-c+c\omega}$; $Z'=\frac{a+c\omega}{b+c\omega}$; &c.

Hinc ſi illius ſeriei terminus indici ω reſpondens ponatur $=\Sigma$, erit ex §. 398:

$$\Sigma=\frac{a(b+c\omega)}{b(a+c\omega)}\cdot\frac{(a+c)(b+c+c\omega)}{(b+c)(a+c+c\omega)}\cdot\frac{(a+2c)(b+2c+c\omega)}{(b+2c)(b+2c+c\omega)}.\ \&c.$$

Quare ſi deſideretur terminus indici $\frac{1}{2}$ reſpondens, facto $\omega=\frac{1}{2}$, erit:

$$\Sigma=\frac{a(2b+c)}{b(2a+c)}\cdot\frac{(a+c)(2b+3c)}{(b+c)(2a+3c)}\cdot\frac{(a+2c)(2b+5c)}{(b+2c)(2a+5c)}.\ \&c.$$

EXEMPLUM.

Interpolare hanc ſeriem:

$$\overset{1}{\frac{1}{2}};\ \overset{2}{\frac{1.3}{2.4}};\ \overset{3}{\frac{1.3.5}{2.4.6}};\ \overset{4}{\frac{1.3.5.7}{2.4.6.8}};\ \overset{5}{\frac{1.3.5.7.9}{2.4.6.8.10}};\ \&c.$$

Cum hic ſit $a=1$, $b=2$, & $c=2$; ſit terminus indici cuicunque ω reſpondens $=\Sigma$, erit

$$\Sigma=\frac{1(2+2\omega)}{2(1+2\omega)}\cdot\frac{3(4+2\omega)}{4(3+2\omega)}\cdot\frac{5(6+2\omega)}{6(5+2\omega)}\cdot\frac{7(8+2\omega)}{8(7+2\omega)}.\ \&c.$$

Hinc ſi termini, qui indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. reſpondent, ponantur Σ', Σ'', Σ''', &c. erit:

$\Sigma'=$

$$\Sigma' = \frac{1+2\omega}{2+2\omega} \cdot \Sigma$$

$$\Sigma'' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \Sigma$$

$$\Sigma''' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \frac{5+2\omega}{6+2\omega} \cdot \Sigma \qquad \&c.$$

Si itaque deſideretur terminus indici $\frac{1}{2}$ reſpondens, facto $\omega = \frac{1}{2}$, erit:

$$\Sigma = \frac{1\cdot 3}{2.2} \cdot \frac{3\cdot 5}{4.4} \cdot \frac{5\cdot 7}{6.6} \cdot \frac{7\cdot 9}{8.8} \cdot \frac{9\cdot 11}{10.10} \cdot \&c.$$

Verum poſito $\pi =$ ſemicircumferentiae circuli, cuius radius eſt $= 1$, ſupra oſtendimus eſſe:

$$\pi = 2 \cdot \frac{2.2}{1\cdot 3} \cdot \frac{4.4}{3\cdot 5} \cdot \frac{6.6}{5\cdot 7} \cdot \frac{8.8}{7\cdot 9} \cdot \&c.$$

Hancobrem termini intermedii indicibus $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. per peripheriam circuli exprimi poterunt, hoc modo:

Indices: $\frac{1}{2}$ $\quad$ $\frac{3}{2}$ $\quad$ $\frac{5}{2}$ $\quad$ $\frac{7}{2}$

Termini: $\frac{2}{\pi}$; $\frac{2}{3} \cdot \frac{2}{\pi}$; $\frac{2\cdot 4}{3\cdot 5} \cdot \frac{2}{\pi}$; $\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} \cdot \frac{2}{\pi}$; &c.

Quam eandem interpolationem *Wallisius* in arithmetica infinitorum invenit:

401. Conſideremus nunc iſtam ſeriem:

$$\overset{1}{a};\ \overset{2}{a(a+b)};\ \overset{3}{a(a+b)(a+2b)};\ \overset{4}{a(a+b)(a+2b)(a+3b)};\ \&c.$$

cuius factores hanc progreſſionem arithmeticam conſtituunt:

$$a,\ (a+b),\ (a+2b),\ (a+3b),\ (a+4b),\ \&c.$$

huiusque termini infiniteſimi ita ſunt comparati, ut eorum logarithmorum differentiae evaneſcant. Cum igitur ſit $Z = a - b + b\omega$, & $Z' = a + b\omega$; $Z'' = a + b + b\omega$; $Z''' = a + 2b + b\omega$; &c. ſi Σ denotet terminum ſeriei propoſitae, cuius index eſt $= \omega$, erit:

$$\Sigma = a^\omega \cdot \frac{a^{1-\omega}(a+b)^\omega}{a+b\omega} \cdot \frac{(a+b)^{1-\omega}(a+2b)^\omega}{a+b+b\omega} \cdot \frac{(a+2b)^{1-\omega}(a+3b)^\omega}{a+2b+b\omega} \cdot \&c.$$

Hocque valore invento, ſi ω denotet numerum quemvis fractum unitate minorem, termini ſequentes indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. reſpondentes ita determinabuntur, ut ſit

$$\Sigma' = (a+b\omega)\ \Sigma$$
$$\Sigma'' = (a+b\omega)\ (a+b+b\omega)\ \Sigma$$
$$\Sigma''' = (a+b\omega)\ (a+b+b\omega)\ (a+2b+b\omega)\ \Sigma$$
&c.

Quare ſi deſideretur terminus indici $\frac{1}{2}$ reſpondens, facto $\omega = \frac{1}{2}$, erit:

$$\Sigma = a^{\frac{1}{2}}.\frac{a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a+\frac{1}{2}b}.\frac{(a+b)^{\frac{1}{2}}(a+2b)^{\frac{1}{2}}}{a+\frac{3}{2}b}.\frac{(a+2b)^{\frac{1}{2}}(a+3b)^{\frac{1}{2}}}{a+\frac{5}{2}b}.\ \&c.$$

ideoque ſumtis quadratis:

$$\Sigma^2 = a.\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)}.\frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)}.\frac{(a+2b)(a+3b)}{(a+\frac{5}{2}b)(a+\frac{5}{2}b)}.\ \&c.$$

402. Ponatur in ſerie quam ſupra tractavimus:

$$\overset{1}{\frac{f}{g}};\ \overset{2}{\frac{f(f+b)}{g(g+b)}};\ \overset{3}{\frac{f(f+b)(f+2b)}{g(g+b)(g+2b)}};\ \overset{4}{\frac{f(f+b)(f+2b)(f+3b)}{g(g+b)(g+2b)(g+3b)}};\ \&c.$$

terminus indici $\frac{1}{2}$ reſpondens $= \Theta$, erit:

$$\Theta = \frac{f(g+\frac{1}{2}b)}{g(f+\frac{1}{2}b)}.\frac{(f+b)(g+\frac{3}{2}b)}{(g+b)(f+\frac{3}{2}b)}.\frac{(f+2b)(g+\frac{5}{2}b)}{(g+2b)(f+\frac{5}{2}b)}.\ \&c.$$

ſtatuatur nunc: $f = a$; $g = a+\frac{1}{2}b$; & $b = b$; erit

$$\Theta = \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)}.\frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)}.\ \&c.$$

ideoque fiet $\Sigma^2 = a\Theta$, & $\Sigma = \sqrt{a\Theta}$. Quocirca ſi huius ſeriei:

$$\overset{1}{a};\ \overset{2}{a(a+b)};\ \overset{3}{a(a+b)(a+2b)};\ \overset{4}{a(a+b)(a+2b)(a+3b)};\ \&c.$$

terminus indici $\frac{1}{2}$ reſpondens ſtatuatur $= \Sigma$; atque huius ſeriei:

a

$$\overset{1}{\frac{a}{a+\frac{1}{2}b}};\ \overset{2}{\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)}};\ \overset{3}{\frac{a(a+b)(a+2b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)(a+\frac{5}{2}b)}};\ \text{\&c.}$$

terminus indici $\frac{1}{2}$ reſpondens ponatur $=\Theta$; erit $\Sigma=\sqrt{a\Theta}$.

Cum igitur hic ſeriei ſolorum numeratorum terminus indici $\frac{1}{2}$ reſpondens ſit $=\Sigma$, ſi in ſerie denominatorum terminus indici $\frac{1}{2}$ reſpondens ponatur $=\Lambda$; erit $\Theta=\frac{\Sigma}{\Lambda}$; at eſt $\Theta=\frac{\Sigma^2}{a}$, unde fiet $\Sigma=\frac{a}{\Lambda}$, ſeu $\Sigma\Lambda=a$, quibus theorematibus interpolatio huiuſmodi ſerierum non mediocriter illuſtratur.

EXEMPLUM I.

Sit propoſita haec ſeries interpolanda:

$$1,\ 1\cdot2,\ 1\cdot2\cdot3,\ 1\cdot2\cdot3\cdot4,\ \text{\&c.}$$

Quia hic eſt $a=1$, & $b=1$, ſi terminus indici ω reſpondens ponatur $=\Sigma$, erit:

$$\Sigma=\frac{1^{1-\omega}2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}4^{\omega}}{3+\omega}\cdot\frac{4^{1-\omega}5^{\omega}}{4+\omega}\ \text{\&c.}$$

Hic pro ω ſemper fractio unitate minor accipi poteſt nihilominus enim interpolatio per totam ſeriem extendetur. Nam ſi termini indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. reſpondentes ponantur Σ', Σ'', Σ''', &c. erit:

$$\Sigma'=(1+\omega)\Sigma$$
$$\Sigma''=(1+\omega)(2+\omega)\Sigma$$
$$\Sigma'''=(1+\omega)(2+\omega)(3+\omega)\Sigma$$
&c.

Seriei ergo propoſitae terminus indici $\frac{1}{2}$ reſpondens erit:

$$\Sigma=\frac{1^{\frac{1}{2}}\cdot2^{\frac{1}{2}}}{1\frac{1}{2}};\ \frac{2^{\frac{1}{2}}\cdot3^{\frac{1}{2}}}{2\frac{1}{2}};\ \frac{3^{\frac{1}{2}}\cdot4^{\frac{1}{2}}}{3\frac{1}{2}};\ \text{\&c.}\quad \text{ſive}$$

$$\Sigma^2=\frac{2\cdot4}{3\cdot3}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{6\cdot8}{7\cdot7}\cdot\frac{8\cdot10}{9\cdot9}\cdot\ \text{\&c.}$$

Un-

Unde cum ſit $\pi = 2 \cdot \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot$ &c.

erit $\Sigma^2 = \frac{\pi}{4}$ & $\Sigma = \frac{\sqrt{\pi}}{2}$: hincque reſpondebunt

Indicibus : $\frac{1}{2}$. $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$

Termini : $\frac{\sqrt{\pi}}{2}$; $\frac{3}{2} \cdot \frac{\sqrt{\pi}}{2}$; $\frac{3.5}{2.2} \cdot \frac{\sqrt{\pi}}{2}$; $\frac{3.5.7}{2.2.2} \cdot \frac{\sqrt{\pi}}{2}$; &c.

EXEMPLUM II.

Sit propoſita haec ſeries interpolanda :

$\overset{1}{1}$; $\overset{2}{1.3}$; $\overset{3}{1.3.5}$; $\overset{4}{1.3.5.7}$; &c.

Quia hic eſt $a = 1$; $b = 2$; ſi terminus indici ω reſpondens ponatur $= \Sigma$ erit :

$$\Sigma = \frac{1^{1-\omega} 3^{\omega}}{1+2\omega} \cdot \frac{3^{1-\omega} 5^{\omega}}{3+2\omega} \cdot \frac{5^{1-\omega} 7^{\omega}}{5+2\omega} \cdot \text{\&c.}$$

terminique ordine ſequentes ita erunt comparati :

$$\Sigma' = (1+2\omega)\Sigma$$
$$\Sigma'' = (1+2\omega)(3+2\omega)\,\Sigma$$
$$\Sigma''' = (1+2\omega)(3+2\omega)(5+2\omega)\,\Sigma$$
&c.

Si igitur ſeriei propoſitae deſideretur terminus indici $\frac{1}{2}$ reſpondens, iſque vocetur $= \Sigma$, erit :

$$\Sigma = \frac{\sqrt{1.3}}{2} \cdot \frac{\sqrt{3.5}}{4} \cdot \frac{\sqrt{5.7}}{6} \cdot \frac{\sqrt{7.9}}{8} \cdot \text{\&c.} \quad \text{ergo}$$

$$\Sigma^2 = \frac{1.3}{2.2} \cdot \frac{3.5}{4.4} \cdot \frac{5.7}{6.6} \cdot \frac{7.9}{8.8} \cdot \text{\&c.} = \frac{2}{\pi},$$

ideoque habebitur $\Sigma = \sqrt{\frac{2}{\pi}}$. At reſpondebunt

Indicibus : $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; $\frac{7}{2}$ &c.

Termini : $\sqrt{\frac{2}{\pi}}$; $2 \cdot \sqrt{\frac{2}{\pi}}$; $2.4\sqrt{\frac{2}{\pi}}$; $2.4.6\sqrt{\frac{2}{\pi}}$; &c.

Quodſi ergo prior ſeries & haec invicem multiplicentur ut habeatur haec ſeries : 1

$$\overset{1}{1^2};\ \overset{2}{1^2.2.3};\ \overset{3}{1^2.2.3\,.5};\ \overset{4}{1^2.2.3^2.4.5.7};\ \overset{5}{1^2.2.3^2.4.5^2.7.9};\ \&c.$$

cuius terminus indici $\frac{1}{2}$ reſpondens erit $= \frac{\sqrt{\pi}}{2}.\sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{2}}$; quod facile perſpicitur, ſi iſti ſeriei haec forma tribuatur:

$$\overset{1}{\frac{1.2}{2}};\ \overset{2}{\frac{1.2.3.4}{2^2}};\ \overset{3}{\frac{1.2.3.4.5.6}{2^3}};\ \overset{4}{\frac{1.2.3.4.5.6.7.8}{2^4}};\ \&c.$$

cuius terminus indici $\frac{1}{2}$ reſpondens manifeſto eſt $= \frac{1}{\sqrt{2}}$.

EXEMPLUM III.

Sit iſta ſeries propoſita interpolanda:

$$\overset{1}{\frac{n}{1}};\ \overset{2}{\frac{n(n-1)}{1.2}};\ \overset{3}{\frac{n(n-1)(n-2)}{1.2.3}};\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}};\ \&c.$$

Conſiderentur huius ſeriei numeratores ac denominatores ſeorſim, & cum numeratores ſint:

$$\overset{1}{n};\ \overset{2}{n(n-1)};\ \overset{3}{n(n-1)(n-2)};\ \overset{4}{n(n-1)(n-2)(n-3)};\ \&c.$$

fiet applicatione facta, $a = n$, & $b = -1$, unde huius ſeriei terminus indici ω reſpondens erit $=$

$$n^{\omega}.\frac{n^{1-\omega}(n-1)^{\omega}}{n-\omega}.\frac{(n-1)^{1-\omega}(n-2)^{\omega}}{n-1-\omega}.\frac{(n-2)^{1-\omega}(n-3)^{\omega}}{n-2-\omega}.\ \&c.$$

quae autem expreſſio ob factores in negativos abeuntes nihil certi monſtrat. Transformetur ergo ſeries propoſita, ponendo brevitatis gratia $1.2.3\ldots n = N$, in hanc:

$$N\left(\overset{1}{\frac{1}{1.1.2.3\ldots(n-1)}};\ \overset{2}{\frac{1}{1.2.1.2.3\ldots(n-2)}};\ \overset{3}{\frac{1}{1.2.3.1.2.3\ldots(n-3)}};\ \&c.\right)$$

cuius denominatores cum conſtent duobus factoribus, alteri conſtituent hanc ſeriem:

$$\overset{1}{1.2.3\ldots(n-1)};\ \overset{2}{1.2.3\ldots(n-2)};\ \overset{3}{1.2.3\ldots(n-3)};\ \&c.$$

 cuius

cuius terminus indici ω reſpondens, convenit cum termino huius ſeriei:

$$\overset{1}{1}\,;\ \overset{2}{1.2}\,;\ \overset{3}{1.2.3}\,;\ \overset{4}{1.2.3.4}\,;\ \overset{5}{1.2.3.4.5}\,;\ \&c.$$

indici $n-\omega$ reſpondente: qui eſt

$$\frac{1^{1-n+\omega}2^{n-\omega}}{1+n-\omega}\cdot\frac{2^{1-n+\omega}3^{n-\omega}}{2+n-\omega}\cdot\frac{3^{1-n+\omega}4^{n-\omega}}{3+n-\omega}.\&c.$$

Sit autem huius ſeriei terminus indici $1-\omega$ reſpondens $=\Theta$; erit:

$$\Theta=\frac{1^{\omega}.2^{1-\omega}}{2-\omega}\cdot\frac{2^{\omega}.3^{1-\omega}}{3-\omega}\cdot\frac{3^{\omega}.4^{1-\omega}}{4-\omega}.\ \&c.$$

atque cum reſpondeant:

Indicibus: $1-\omega$; $2-\omega$; $3-\omega$

Termini: Θ; $(2-\omega)\Theta$; $(2-\omega)(3-\omega)\Theta$; &c.

indici $n-\omega$ reſpondebit hic terminus:

$$(2-\omega)(3-\omega)(4-\omega)\ \ldots\ (n-\omega)\Theta.$$

Deinde illorum denominatorum alteri factores conſtituent hanc ſeriem:

$$\overset{1}{1}\,;\ \overset{2}{1.2}\,;\ \overset{3}{1.2.3}\,;\ \overset{4}{1.2.3.4}\,;\ \overset{5}{1.2.3.4.5}\,;\ \&c.$$

ſi terminus indici ω reſpondens ponatur $=\Lambda$, erit:

$$\Lambda=\frac{1^{1-\omega}2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}4^{\omega}}{3+\omega}.\ \&c.$$

Quibus inventis ſi ipſius ſeriei propoſitae:

$$\overset{1}{\frac{n}{1}}\,;\ \overset{2}{\frac{n(n-1)}{1.2}}\,;\ \overset{3}{\frac{n(n-1)(n-2)}{1.2.3}}\,;\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}}\,;\ \&c.$$

terminus indici ω reſpondens ponatur Σ, erit:

$$\Sigma=\frac{N}{\Lambda.(2-\omega)(3-\omega)(4-\omega)\ \ldots\ (n-\omega)\Theta}.$$

At vero eſt:

$$\frac{N}{(2-\omega)(3-\omega)(4-\omega)\ \ldots\ (n-\omega)}=$$

2

$$\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdot\;\ldots\;\cdot\frac{n}{n-\omega};$$

atque

$$\Lambda\Theta=\frac{1.2}{(1+\omega)(2-\omega)}\cdot\frac{2.3}{(2+\omega)(3-\omega)}\cdot\frac{3.4}{(3+\omega)(4-\omega)}\cdot\&c.$$

Ex quibus terminus indici ω reſpondens quaeſitus erit:

$$\Sigma=\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdot\frac{5}{5-\omega}\cdot\;\ldots\;\cdot\frac{n}{n-\omega}\cdot$$

$$\frac{(1+\omega)(2-\omega)}{1.\;2}\cdot\frac{(2+\omega)(3-\omega)}{2.\;3}\cdot\frac{(3+\omega)(4-\omega)}{3.\;4}.\&c.\text{ in infinitum.}$$

Indici ergo $\frac{1}{2}$ reſpondebit iſte terminus.

$$\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9}\cdot\frac{12}{11}\;\ldots\;\cdot\frac{2n}{2n-1}\cdot$$

$$\frac{3.3}{2.4}\cdot\frac{5.5}{4.6}\cdot\frac{7.7}{6.8}\cdot\frac{9.9}{8.10}.\&c.$$

qui reducitur ad $\frac{4.6.8.10\;\ldots\;2n}{3.5.7.9\;\ldots\;(2n-1)}\cdot\frac{4}{\pi}$, ſeu

erit $=\frac{2}{\pi}\cdot\frac{2.4.6.8.10\;\ldots\;2n}{1.3.5.7.9\;\ldots\;(2n-1)}$

Si fuerit $n=2$, prodibit iſta ſeries interpolanda:

0,	1,	2,	3,	4,	5,	6,	&c.
1,	2,	1,	0,	0,	0,	0,	&c.

cuius propterea terminus indici $\frac{1}{2}$ reſpondens eſt $=\frac{16}{3\pi}$.

EXEMPLUM IV.

Quaeratur terminus reſpondens indici $=\frac{1}{2}$ in hac ſerie:

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & +\frac{1}{2} & -\frac{1.1}{2.4} & +\frac{1.1.3}{2.4.6} & -\frac{1.1.3.5}{2.4.6.8} \end{matrix}+\&c.$$

Oritur haec ſeries ex praecedente ſi ponatur $n=\frac{1}{2}$, eritque propterea terminus quaeſitus, qui ſit $=\Sigma$:

$$\Sigma = \frac{2}{\pi} \cdot \frac{2.4.6.8 \quad . \quad . \quad . \quad . \quad 2n}{1.3.5.7 \quad . \quad . \quad . \quad (2n-1)} \text{ poſito } n = \tfrac{1}{2}.$$

Ponatur $\frac{2.4.6.8 \quad . \quad . \quad . \quad . \quad 2n}{1.3.5.7 \quad . \quad . \quad . \quad (2n-1)} = \Theta$ ſi ſit $n = \frac{1}{2}$, eritque Θ terminus reſpondens indici $\frac{1}{2}$ in hac ſerie:

$$\frac{2}{1};\ \frac{2.4}{1.3};\ \frac{2.4.6}{1.3.5};\ \frac{2.4.6.8}{1.3.5.7};\ \&c.$$

qui ex ſuperioribus prodit $= \frac{\pi}{2}$. Quocirca ſeriei propoſitae terminus indici $\frac{1}{2}$ reſpondens, qui quaeritur, erit $= 1$. Quoniam autem in iſta ſerie, ſi terminus indici cuicunque ω reſpondens ponatur $= \Sigma$, ſequens eum erit $\Sigma' = \frac{1-2\omega}{2+2\omega}\Sigma$; ſeries propoſita ita mediis terminis interiiciendis interpolabitur:

Indices:	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	
Termini:	1;	1;	$\frac{1}{2}$;	0;	$\frac{-1.1}{2.4}$;	0;	$\frac{1.1.3}{2.4.6}$;	0;	&c.

EXEMPLUM V.

Si n *fuerit numerus quicunque fractus, invenire terminum indici* ω *reſpondentem in ſerie:*

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1; & \frac{n}{1}; & \frac{n(n-1)}{1.\ 2}; & \frac{n(n-1)(n-2)}{1.\ 2.\ 3}; & \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} \end{matrix}$$

&c.

Si expreſſionem $\frac{2}{2-\omega} \cdot \frac{3}{3-\omega} \cdot \frac{4}{4-\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad \frac{n}{n-\omega}$ cum §. 400. comparemus, fiat $a = 1$, $c = 1$, $b = 1 - \omega$, ibique loco ω poſito n, erit:

$$\frac{1}{1-\omega} \cdot \frac{2}{2-\omega} \cdot \frac{3}{3-\omega} \cdot \; . \; . \; . \frac{n}{n-\omega} = \frac{1(1-\omega+n)}{(1-\omega)(1+n)} \cdot \frac{2(2-\omega+n)}{(2-\omega)(2+n)} \cdot \&c.$$

unde

unde terminus quaeſitus indici ω reſpondens ſi ponatur $= \Sigma$, erit:

$$\Sigma = \frac{(1-\omega+n)2}{(1+n)(2-\omega)} \cdot \frac{(2-\omega+n)3}{(2+n)(3-\omega)} \cdot \&c. \frac{(1+\omega)(2-\omega)}{1. \quad 2} \cdot \frac{(2+\omega)(3-\omega)}{2. \quad 3} \cdot \&c.$$

ideoque

$$\Sigma = \frac{(1+\omega)(1+n-\omega)}{1(1+n)} \cdot \frac{(2+\omega)(2+n-\omega)}{2(2+n)} \cdot \frac{(3+\omega)(3+n-\omega)}{3(3+n)} \cdot \&c.$$

quoties ergo $n - \omega$ fuerit numerus integer valor ipſius Σ rationaliter exprimi poteſt.

Sic ſi ſit $n = \omega$ erit $\Sigma = 1$

ſi $n = 1 + \omega$ erit $\Sigma = n$

ſi $n = 2 + \omega$ erit $\Sigma = \frac{n(n-1)}{1. \ 2}$

ſi $n = 3 + \omega$ erit $\Sigma = \frac{n(n-1)(n-2)}{1. \ 2. \ 3}$

&c.

At ſi fuerit $\omega - n$ numerus integer affirmativus, erit ſemper $\Sigma = 0$.

CAPUT XVIII.

DE USU CALCULI DIFFERENTIALIS IN RESOLUTIONE FRACTIONUM.

403.

Methodus fractionem quamvis propoſitam in fractiones ſimplices reſolvendi, quam in Introductione expoſuimus, etſi per ſe ſatis eſt facilis; tamen ope calculi differentialis ita perfici poteſt, ut ſaepenumero multo minori negotio in uſum vocari poſſit. Praecipue vero ſi denominator fractionis reſolvendae fuerit indefiniti gradus, methodus ante expoſita plerumque non mediocriter impeditur, dum loco quantitatis incognitae ſubſtitutio valoris, quem ex quopiam factore induit, fieri debet. Imprimis autem his caſibus diviſio denominatoris per factorem iam inventum nimis fit moleſta. Quae operatio, ſi calculus differentialis in ſubſidium vocetur, evitari poterit, ita ut non opus ſit alterum denominatoris factorem, qui oritur, ſi denominator per factorem iam cognitum dividatur, noſſe. Hunc autem uſum praeſtat methodus determinandi valorem fractionis, cuius numerator ac denominator certo caſu ambo evaneſcunt, cuius beneficio, quemadmodum reſolutio fractionum iam ſupra tradita commodior & tractabilior reddi queat, hoc Capite doceamus, ſimulque finem huic libro, in quo uſum calculi differentialis in Analyſi expoſuimus, imponamus.

404. Si igitur propoſita fuerit fractio quaecunque $\frac{P}{Q}$; cuius numerator ac denominator ſint functiones variabilis quantitatis x, rationales & integrae; primum videndum eſt, utrum x in numeratore P tot pluresve dimenſiones habeat, quam in deno-

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denominatore Q. Quodſi eveniat, complectetur fractio $\frac{P}{Q}$ in ſe partem integram huius formae $A + Bx + Cx^2 + \&c.$ quae diviſionis ope inde erui poterit: pars reliqua erit fractio eundem denominatorem Q habens, ſed cuius numerator erit functio puta R pauciores ipſius x dimenſiones continens, quam denominator Q, ita ut ulterior reſolutio inſtituenda ſit in fractione $\frac{R}{Q}$. Interim tamen non opus eſt noſſe hunc novum numeratorem R, ſed eaedem fractiones ſimplices, quas fractio $\frac{R}{Q}$ ſuppeditatura eſſet, elici poſſunt immediate ex fractione propoſita $\frac{P}{Q}$; prouti iam ſupra notavimus.

405. Praeter partem integram igitur, ſi quam continet fractio $\frac{P}{Q}$, erui debent fractiones ſimplices, quarum denominatores ſint vel binomiales huius formae $f+gx$, vel trinomiales huiuſmodi $f+2x\cos\varphi.\sqrt{fg}+gxx$, vel eiuſmodi formularum quadrata, cubive ſeu altiores poteſtates. Hique denominatores omnes erunt factores denominatoris Q, ita ut quilibet denominatoris ipſius Q factor praebeat fractionem ſimplicem. Scilicet ſi denominator Q factorem habeat $f+gx$, ex eo naſcetur fractio ſimplex huiuſmodi $\frac{\mathfrak{A}}{f+gx}$; ſin autem factor fuerit $(f+gx)^2$, binae fractiones $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$. Atque ex denominatoris Q factore cubico $(f+gx)^3$ oriuntur tres fractiones ſimplices huius formae:
$\frac{\mathfrak{A}}{(f+gx)^3} + \frac{\mathfrak{B}}{(f+gx)^2} + \frac{\mathfrak{C}}{f+gx}$; & ita porro. Quodſi autem denominator Q factorem habuerit trinomialem huiuſmodi

iuſmodi $ff - 2fgx\cos\varphi + ggxx$, ex eo orietur fractio ſimplex talis formae $\frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\varphi + ggxx}$; &, ſi duo huiuſmodi factores fuerint aequales uti $(ff - 2fgx\cos\varphi + ggxx)^2$, hinc prodibunt duae fractiones

$$\frac{\mathfrak{A} + \mathfrak{a}x}{(ff - 2fgx\cos\varphi + ggxx)^2} + \frac{\mathfrak{B} + \mathfrak{b}x}{ff - 2fgx\cos\varphi + ggxx}.$$

Huiuſmodi autem factor cubicus $(ff - 2fgx\cos\varphi + ggxx)^3$ dabit tres fractiones ſimplices, biquadratus quatuor, & ita porro.

406. Reſolutio ergo fractionis cuiuſcunque $\frac{P}{Q}$ ita inſtituatur. Quaerantur primo omnes factores tam ſimplices ſeu binomiales, quam trinomiales denominatoris Q, & ſi qui fuerint inter ſe aequales, ii probe notentur, & inſtar unius habeantur. Tum ex ſingulis his denominatoris factoribus eliciantur fractiones ſimplices, vel modo iam ſupra oſtenſo, vel eo, quem hic ſumus traditurí, & qui pro lubitu in locum prioris ſubſtitui poterit. Quo facto aggregatum omnium iſtarum fractionum ſimplicium una cum parte integra, ſi quam continet fractio propoſita $\frac{P}{Q}$, huius valorem exhaurient. Inventionem quidem factorum denominatoris Q hic tanquam cognitam aſſumimus, cum pendeat a reſolutione aequationis $Q = 0$; methodumque hic trademus per calculum differentialem pro dato quovis denominatoris factore fractionem ſimplicem inde ortam definiendi. Quod, cum iſtarum fractionum ſimplicium denominatores iam habeantur, praeſtabitur, ſi numeratorem cuiuſque fractionis inveſtigare doceamus.

407. Ponamus ergo fractionis $\frac{P}{Q}$ denominatorem Q factorem habere $f + gx$, ita ut ſit $Q = (f + gx)S$ neque vero hic alter factor S inſuper eundem factorem $f + gx$ contineat.

Sit

Sit fractio ſimplex ex iſto factore orta $=\frac{\mathfrak{A}}{f+gx}$; & complementum huiuſmodi formam habebit $\frac{V}{S}$, ita ut ſit $\frac{\mathfrak{A}}{f+gx}+\frac{V}{S}=\frac{P}{Q}$. Erit ergo $\frac{V}{S}=\frac{P}{Q}-\frac{\mathfrak{A}}{f+gx}=\frac{P-\mathfrak{A}S}{(f+gx)S}$; ideoque $V=\frac{P-\mathfrak{A}S}{f+gx}$. Cum igitur V ſit functio integra ipſius x, neceſſe eſt ut $P-\mathfrak{A}S$ ſit diviſibile per $f+gx$; ac propterea ſi ponatur $f+gx=0$ ſeu $x=\frac{-f}{g}$, expreſſio $P-\mathfrak{A}S$ evaneſcet. Fiat ergo $x=\frac{-f}{g}$, & cum ſit $P-\mathfrak{A}S=0$, erit $\mathfrak{A}=\frac{P}{S}$, uti iam ſupra invenimus. Cum autem ſit $S=\frac{Q}{f+gx}$, fiet $\mathfrak{A}=\frac{(f+gx)P}{Q}$, ſi ubique ponatur $f+gx=0$, ſeu $x=\frac{-f}{g}$. Quoniam vero hoc caſu tam numerator $(f+gx)P$, quam denominator Q evaneſcit; per ea, quae de valore huiuſmodi fractionum inveſtigando expoſuimus, erit $\mathfrak{A}=\frac{(f+gx)dP+Pgdx}{dQ}$, ſi quidem ponatur $x=\frac{-f}{g}$. Hoc autem caſu ob $(f+gx)dP=0$, erit $\mathfrak{A}=\frac{gPdx}{dQ}$; ſicque per differentiationem valor numeratoris $\mathfrak{A}$ expedite reperitur.

408. Si igitur fractionis propoſitae $\frac{P}{Q}$ denominator Q factorem habeat ſimplicem $f+gx$, ex eo orietur fractio ſim-

ſimplex $\frac{\mathfrak{A}}{f+gx}$, exiſtente $\mathfrak{A} = \frac{gPdx}{dQ}$, poſtquam hic ubique loco x valor $\frac{-f}{g}$ ex aequatione $f+gx=0$ oriundus fuerit ſubſtitutus. Hoc ergo modo non neceſſe eſt, ut ante quaeratur alter denominatoris Q factor S, qui oritur, ſi Q per $f+gx$ dividatur. Hinc ſi Q non in factoribus exprimatur, hanc diviſionem ſaepe non parum moleſtam, praecipue ſi x in denominatore Q habeat exponentes indefinitos omittere poterimus, cum valor ipſius $\mathfrak{A}$ ex formula $\frac{gPdx}{dQ}$ obtineatur. Sin autem denominator Q iam in factoribus fuerit expreſſus, ita ut inde valor ipſius S ſponte pateat, tum praeferenda erit altera expreſſio, qua invenimus $\mathfrak{A} = \frac{P}{S}$, ponendo pariter ubique $x = \frac{-f}{g}$. Sicque pro inveniendo valore ipſius $\mathfrak{A}$ quovis caſu ea formula adhiberi poterit, quae commodior & expeditior videatur. Uſum autem novae formulae aliquot exemplis illuſtrabimus.

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{x^9}{1+x^{17}}$, *cuius fractionem ſimplicem ex denominatoris factore* $1+x$ *oriundam definiri oporteat.*

Quoniam hic eſt $Q = 1 + x^{17}$, cuius etſi factor $1 + x$ conſtat, tamen ſi, uti prima methodus poſtulat per eum dividere velimus, prodiret

$$S = 1 - x + xx - x^3 + \ldots + x^{16}.$$

Commodius igitur utemur nova formula $\mathfrak{A} = \frac{gPdx}{dQ}$; quia itaque eſt $f=1$, $g=1$, & $P=x^9$, ob $dQ = 17x^{16}dx$; fiet

fiet $\mathfrak{A} = \frac{x^9}{17x^{16}} = \frac{1}{17x^7}$, poſito $x = -1$, unde fit $\mathfrak{A} = -\frac{1}{17}$, & fractio ſimplex ex denominatoris factore $1+x$ oriunda erit $\frac{-1}{17(1+x)}$.

EXEMPLUM II.

Propoſita fractione $\frac{x^m}{1-x^{2n}}$ *fractionem ſimplicem ex denominatoris factore* $1-x$ *oriundam inveſtigare.*

Ob factorem propoſitum $1-x$, erit $f = 1$, & $g = -1$. Tum vero denominator $Q = 1 - x^{2n}$ dat $dQ = -2nx^{2n-1}dx$; unde propter $P = x^m$ obtinebitur $\mathfrak{A} = \frac{-x^m}{-2nx^{2n-1}}$. Poſitoque ex aequatione $1-x=0$, $x=1$, fiet $\mathfrak{A} = \frac{1}{2n}$; ita ut fractio ſimplex futura ſit haec $\frac{1}{2n(1-x)}$.

EXEMPLUM III.

Propoſita fractione $\frac{x^m}{1-4x^k+3x^n}$, *eius fractionem ſimplicem ex denominatoris factore* $1-x$ *oriundam determinare.*

Hic ergo fit $f=1$; $g=-1$; $P=x^m$; $Q=1-4x^k+3x^n$ & $\frac{dQ}{dx} = -4kx^{k-1}+3nx^{n-1}$; unde fit $\mathfrak{A} = \frac{-x^m}{-4kx^{k-1}+3nx^{n-1}}$ & poſito $x=1$, erit $\mathfrak{A} = \frac{1}{4k-3n}$. Fractio ergo ſimplex ex iſto denominatoris factore ſimplici $1-x$ oriunda erit $= \frac{1}{(4k-3n)(1-x)}$.

409. Ponamus nunc fractionis $\frac{P}{Q}$ denominatorem Q factorem habere quadratum $(f+gx)^2$, & fractiones ſimplices hinc oriundas eſſe $= \frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$. Sit $Q = (f+gx)^2 S$ & complementum $= \frac{V}{S}$; ita ut ſit $\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{(f+gx)^2} - \frac{\mathfrak{B}}{f+gx}$; & $V = \frac{P - \mathfrak{A}S - \mathfrak{B}(f+gx)S}{(f+gx)^2}$. Quia nunc V eſt functio integra, neceſſe eſt ut ſit $P - \mathfrak{A}S - \mathfrak{B}S(f+gx)$ diviſibile per $(f+gx)^2$; & cum S factorem $f+gx$ amplius non contineat, quoque haec expreſſio $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx)$ diviſibilis erit per $(f+gx)^2$; ideoque facto $f+gx=0$, ſeu $x = \frac{-f}{g}$ non ſolum ipſa, ſed etiam eius differentiale $d.\frac{P}{S} - \mathfrak{B}gdx$ evaneſcet. Fiat ergo $x = \frac{-f}{g}$, eritque ex priori aequatione $\mathfrak{A} = \frac{P}{S}$; ex poſteriori vero erit $\mathfrak{B} = \frac{1}{gdx}\, d.\frac{P}{S}$; quibus valoribus inventis habebuntur fractiones quaeſitae: $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$.

EXEMPLUM.

Propoſita fractione $\frac{x^m}{1-4x^3+3x^4}$ *cujus denominator factorem habet* $(1-x)^2$, *invenire fractiones ſimplices hinc oriundas.*

Cum hic ſit $f=1$, $g=-1$, $P=x^m$ & $Q=1-4x^3+3x^4$, erit $S = 1+2x+3xx$; $\frac{P}{S} = \frac{x^m}{1+2x+3xx}$; &

d.

$$d.\ \frac{P}{S} = \frac{mx^{m-1}dx + 2(m-1)x^m dx + 3(m-2)x^{m+1}dx}{(1+2x+3xx)^2}.$$

Hinc poſito $x = 1$, erit:

$$\mathfrak{A} = \frac{1}{6} \ \&\ \mathfrak{B} = -1 \cdot \frac{6m-8}{36} = \frac{4-3m}{18};$$

unde fractiones quaeſitae erunt: $\dfrac{1}{6(1-x)^2} + \dfrac{4-3m}{18(1-x)}$.

410. Habeat fractionis $\frac{P}{Q}$ denominator Q tres factores ſimplices aequales, ſeu ſit $Q = (f+gx)^3 S$, ſintque fractiones ſimplices ex hoc factore cubico $(f+gx)^3$ oriundae hae:

$$\frac{\mathfrak{A}}{(f+gx)^3} + \frac{\mathfrak{B}}{(f+gx)^2} + \frac{\mathfrak{C}}{f+gx};$$

complementum vero harum fractionum ad fractionem propoſitam $\frac{P}{Q}$ conſtituendam ſit $\frac{V}{S}$, eritque $V = \dfrac{P - \mathfrak{A}S - \mathfrak{B}S(f+gx) - \mathfrak{C}S(f+gx)^2}{(f+gx)^3}$. Quare haec expreſſio $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2$ diviſibilis erit per $(f+gx)^3$; unde poſito $f+gx = 0$ ſeu $x = \frac{-f}{g}$, non ſolum ipſa haec expreſſio, ſed etiam eius differentiale primum & ſecundum evadet $= 0$. Erit ſcilicet ponendo $x = \frac{-f}{g}$:

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 = 0$$

$$d.\ \frac{P}{S} - \mathfrak{B}g\,dx - 2\mathfrak{C}g\,dx\,(f+gx) = 0$$

$$dd.\ \frac{P}{S} - 2\mathfrak{C}g^2dx^2 = 0.$$

 Ex

Ex prima aequatione ergo erit $\mathfrak{A} = \frac{P}{S}$

Ex ſecunda vero erit $\mathfrak{B} = \frac{1}{g\,dx}\,d.\frac{P}{S}$

Ex tertia denique definitur $\mathfrak{C} = \frac{1}{2g^2\,dx^2}\,dd.\frac{P}{S}$

411. Generaliter ergo ſi fractionis $\frac{P}{S}$ denominator Q factorem habeat $(f+gx)^n$, ita ut ſit $Q=(f+gx)^n S$; poſitis fractionibus ſimplicibus ex hoc factore $(f+gx)^n$ oriundis his:

$$\frac{\mathfrak{A}}{(f+gx)^n}+\frac{\mathfrak{B}}{(f+gx)^{n-1}}+\frac{\mathfrak{C}}{(f+gx)^{n-2}}+\frac{\mathfrak{D}}{(f+gx)^{n-3}}+\frac{\mathfrak{E}}{(f+gx)^{n-4}}+\&c.$$

quoad ad ultimam, cuius denominator eſt $f+gx$, perveniatur, ſi ratiocinium ut ante inſtituatur, reperietur haec expreſſio:

$$\frac{P}{S}-\mathfrak{A}-\mathfrak{B}(f+gx)-\mathfrak{C}(f+gx)^2-\mathfrak{D}(f+gx)^3-\mathfrak{E}(f+gx)^4-\&c.$$

diviſibilis eſſe debere per $(f+gx)^n$, hinc tam ipſa, quam ſingula eius differentialia uſque ad gradum $n-1$, caſu $x=\frac{-f}{g}$ evaneſcere debebunt. Ex quibus aequationibus concludetur fore ponendo ubique $x=\frac{-f}{g}$

$$\mathfrak{A} = \frac{P}{S}$$

$$\mathfrak{B} = \frac{1}{1\,g\,dx}\,d.\frac{P}{S}$$

$$\mathfrak{C} = \frac{1}{1.2\,g^2\,dx^2}\,dd.\frac{P}{S}$$

$$\mathfrak{D} = \frac{1}{1.2.3\,g^3\,dx^3}\,d^3.\frac{P}{S}$$

$$\mathfrak{E} = \frac{1}{1.2.3.4\,g^4\,dx^4}\,d^4.\frac{P}{S} \quad \&c.$$

Ubi

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Ubi quidem notandum eſt, differentialia iſta ipſius $\frac{P}{S}$ ante capi oportere, quam loco x ponatur $\frac{-f}{g}$, alias enim variabilitas ipſius x tolleretur.

412. Facilius ergo hoc modo iſti numeratores $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. exprimuntur, quam eo modo, qui in Introductione eſt traditus, & ſaepenumero quoque hac nova ratione eorum valores expeditius reperiuntur. Quae comparatio quo facilius inſtitui queat, valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. priori modo definiamus:

Poſito $x = \frac{-f}{g}$	Statuatur relicto x variabili.
$\mathfrak{A} = \frac{P}{S}$	$\frac{P - \mathfrak{A}S}{f+gx} = \mathfrak{P}$ erit
$\mathfrak{B} = \frac{\mathfrak{P}}{S}$	$\frac{\mathfrak{P} - \mathfrak{B}S}{f+gx} = \mathfrak{Q}$ erit
$\mathfrak{C} = \frac{\mathfrak{Q}}{S}$	$\frac{\mathfrak{Q} - \mathfrak{C}S}{f+gx} = \mathfrak{R}$ erit
$\mathfrak{D} = \frac{\mathfrak{R}}{S}$	$\frac{\mathfrak{R} - \mathfrak{D}S}{f+gx} = \mathfrak{S}$ erit.
$\mathfrak{E} = \frac{\mathfrak{S}}{S}$	& ita porro.

413. Quodſi autem fractionis $\frac{P}{Q}$ denominator Q non omnes factores ſimplices habeat reales, tum bini imaginariorum iunctim ſumantur, quorum productum erit reale. Sit ergo denominatoris Q factor $ff - 2fgx\,\text{coſ}\,\phi + ggxx$, qui poſitus $= 0$ dat hunc duplicem valorem imaginarium:

$$x = \frac{f}{g}\,\text{coſ}\,\phi \pm \frac{f}{g\sqrt{-1}}\,\text{ſin}\,\phi; \quad \text{ex quo erit}$$

$$x^n = \frac{f^n}{g^n}\,\text{coſ}\,n\phi \pm \frac{f^n}{g^n\sqrt{-1}}\,\text{ſin}.\,n\phi.$$

Pona-

Ponamus eſſe $Q = (ff - 2fgx\cos\varphi + ggxx)S$, atque S praeterea per $ff - 2fgx\cos\varphi + ggxx$ non eſſe diviſibile. Sit fractio ex iſto factore denominatoris oriunda:

$$\frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\varphi + ggxx}$$

& complementum ad propoſitam $\frac{P}{Q}$ ſit $= \frac{V}{S}$, erit $V = \frac{P - (\mathfrak{A} + \mathfrak{a}x)S}{ff - 2fgx\cos\varphi + ggxx}$; unde $P - (\mathfrak{A} + \mathfrak{a}x)S$ ac propterea quoque $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ diviſibile erit per $ff - 2fgx\cos\varphi + ggxx$. Evaneſcet ergo $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ ſi ponatur $ff - 2fgx\cos\varphi + ggxx = 0$, hoc eſt ſi ponatur

$$\text{vel} \quad x = \frac{f}{g}\cos\varphi + \frac{f}{g\sqrt{-1}}\sin\varphi$$

$$\text{vel} \quad x = \frac{f}{g}\cos\varphi - \frac{f}{g\sqrt{-1}}\sin\varphi.$$

414. Quoniam P & S ſunt functiones integrae ipſius x, fiat in utroque ſeorſim utraque ſubſtitutio; & quia pro quavis poteſtate ipſius x, puta x^n binomium hoc $x^n = \frac{f^n}{g^n}\cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}}\sin n\varphi$ ſubſtitui debet. Ponamus primo ubique $\frac{f^n}{g^n}\cos n\varphi$ pro x^n, hocque facto abeat P in $\mathfrak{P}$, & S in $\mathfrak{S}$. Deinde ponatur ubique $\frac{f^n}{g^n}\sin n\varphi$ pro x^n, hocque facto abeat P in $\mathfrak{p}$ & S in $\mathfrak{s}$; ubi notandum eſt ante has ſubſtitutiones utramque functionem P & S penitus debere evolvi, ita ut, ſi forte factoribus ſint implicatae, ii per actualem multiplicationem tollantur. His valoribus $\mathfrak{P}$, $\mathfrak{p}$, $\mathfrak{S}$, $\mathfrak{s}$, inventis, manifeſtum erit, ſi ponatur:

$x =$

$$x = \frac{f}{g}\cos\phi \pm \frac{f}{g\sqrt{-1}}\sin\phi$$

functionem P abituram eſſe in $\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}}$, & functionem S abituram eſſe in $\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$: Hinc cum $\frac{P}{S} = \mathfrak{A} - \mathfrak{a}x$ ſeu $P = (\mathfrak{A} + \mathfrak{a}x)S$ utroque caſu evaneſcere debeat, erit:

$$\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}} = \left(\mathfrak{A} + \frac{\mathfrak{a}f}{g}\cos\phi \pm \frac{\mathfrak{a}f}{g\sqrt{-1}}\sin\phi\right)\left(\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}\right)$$

unde ob ſigna ambigua hae duae aequationes orientur:

$$\mathfrak{P} = \mathfrak{A}\mathfrak{S} + \frac{\mathfrak{a}f\mathfrak{S}}{g}\cos\phi - \frac{\mathfrak{a}f\mathfrak{s}}{g}\sin\phi$$

$$\mathfrak{p} = \mathfrak{A}\mathfrak{s} + \frac{\mathfrak{a}f\mathfrak{s}}{g}\cos\phi + \frac{\mathfrak{a}f\mathfrak{S}}{g}\sin\phi$$

ex quibus eliminando $\mathfrak{A}$ eruitur:

$$\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P} = \frac{\mathfrak{a}f(\mathfrak{S}^2 + \mathfrak{s}^2)}{g}\sin\phi\text{; ideoque erit}$$

$$\mathfrak{a} = \frac{g(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P})}{f(\mathfrak{S}^2 + \mathfrak{s}^2)\sin\phi}.$$

Deinde eliminando $\sin\phi$ erit:

$$\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p} = (\mathfrak{S}^2 + \mathfrak{s}^2)\left(\mathfrak{A} + \frac{\mathfrak{a}f}{g}\cos\phi\right).\quad\text{Ergo}$$

$$\mathfrak{A} = \frac{\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p}}{\mathfrak{S}^2 + \mathfrak{s}^2} - \frac{(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P})\cos\phi}{(\mathfrak{S}^2 + \mathfrak{s}^2)\sin\phi}.$$

415. Cum iam ſit $S = \frac{Q}{ff - 2fgx\cos\phi + ggxx}$; quia poſito $ff - 2fgx\cos\phi + ggxx = 0$ tam numerator quam denominator evaneſcent, erit hoc caſu $S = \frac{dQ:dx}{2ggx - 2fg\cos\phi}$. Ponamus nunc ſi ubique ſubſtituatur $x^n = \frac{f^n}{g^n}\cos n\phi$, functio-

ctionem $\frac{dQ}{dx}$ abire in $\mathfrak{Q}$; ſin autem ſtatuatur $x^n = \frac{f^n}{g^n}\sin n\phi$, eam abire in $\mathfrak{q}$; atque manifeſtum eſt ſi ponatur

$$x = \frac{f}{g}\cos\phi \pm \frac{f}{g\sqrt{-1}}\sin\phi$$

functionem $\frac{dQ}{dx}$ abire in $\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}}$. Ex quo functio S abibit in $\frac{\mathfrak{Q} \pm \mathfrak{q} : \sqrt{-1}}{\pm 2fg\sin\phi : \sqrt{-1}}$. Cum ergo ſit $S = \mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$ eodem valore pro x poſito, habebitur,

$$\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}} = \pm \frac{2fg\mathfrak{S}}{\sqrt{-1}}\sin\phi - 2fg\mathfrak{s}\sin\phi.$$

Erit ergo $\mathfrak{s} = \frac{-\mathfrak{Q}}{2fg\sin\phi}$ & $\mathfrak{S} = \frac{\mathfrak{q}}{2fg\sin\phi}$.

Hiſque valoribus ſubſtitutis, fiet $\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}$

& $\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})\cos\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2}$.

416. Hinc ergo idonea obtinetur ratio ex quovis factore ſecundae poteſtatis fractionem ſimplicem formandi, hicque cum ipſe fractionis propoſitae denominator in computo retineatur, diviſionem, qua valor litterae S definiri deberet, & quae ſaepe non parum eſt moleſta, evitamus. Si igitur fractionis $\frac{P}{Q}$ denominator Q factorem habeat talem $ff - 2fgx\cos\phi + ggxx$, ſequenti modo fractio ſimplex ex hoc factore oriunda, quam fingamus $= \frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\phi + ggxx}$, definietur. Ponatur $x = \frac{f}{g}\cos\phi$, & pro quavis ipſius x poteſtate

x^n

x^n ſcribatur $\frac{f^n}{g^n}\cos n\phi$; quo facto abeat P in $\mathfrak{P}$, & functio $\frac{dQ}{dx}$ in $\mathfrak{Q}$. Deinde ibidem ponatur $x = \frac{f}{g}\sin\phi$, & poteſtas eius quaevis $x^n = \frac{f^n}{g^n}\sin n\phi$; abeatque P in $\mathfrak{p}$, & $\frac{dQ}{dx}$ in $\mathfrak{q}$. Inventiſque hoc modo valoribus litterarum $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{p}$ & $\mathfrak{q}$ quantitates $\mathfrak{A}$ & $\mathfrak{a}$ ita definientur, ut ſit

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})\cos\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

$$\mathfrak{a} = \frac{2gg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}.$$

Fractio ergo ex denominatoris Q factore $ff - 2fgx\cos\phi + ggxx$ oriunda erit:

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\cos\phi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\cos\phi + ggxx)}$$

EXEMPLUM I.

Si propoſita fuerit haec fractio $\frac{x^m}{a + bx^n}$, *cuius denominator* $a + bx^n$ *factorem habeat hunc:* $ff - 2fgx\cos\phi + ggxx$ *invenire fractionem ſimplicem huic factori convenientem.*

Quoniam hic eſt $P = x^m$ & $Q = a + bx^n$, erit $\frac{dQ}{dx} = nbx^{n-1}$, unde fiet:

$$\mathfrak{P} = \frac{f^m}{g^m}\cos m\phi \quad ; \quad \mathfrak{p} = \frac{f^m}{g^m}\sin m\phi$$

$$\mathfrak{Q} = \frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\phi \; ; \; \mathfrak{q} = \frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\phi.$$

Ex his erit: $\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2b^2f^{2(n-1)}}{g^{2(n-1)}}$;

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\ \sin(n-m-1)\phi\ ;$$

atque $$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\ \cos(n-m-1)\phi\,.$$

Quamobrem erit fractio ſimplex quaeſita:

$$\frac{2g^{n-m}[f\sin\phi.\sin(n-m-1)\phi + gx\cos(n-m-1)\phi - f\cos\phi.\cos(n-m-1)\phi]}{nbf^{n-m-1}(ff-2fgx\cos\phi+ggxx)}$$

ſeu

$$\frac{2g^{n-m}[gx\cos(n-m-1)\phi - f\cos(n-m)\phi]}{nbf^{n-m-1}(ff-2fgx\cos\phi+ggxx)}.$$

EXEMPLUM II.

Sit propoſita haec fractio $\frac{1}{x^m(a+bx^n)}$, *cuius denominator factorem habeat* $ff-2fgx\cos\phi+ggxx$, *invenire fractionem ſimplicem inde oriundam.*

Cum ſit $P=1$, & $Q = ax^m + bx^{m+n}$,

erit $\frac{dQ}{dx} = max^{m-1} + (m+n)bx^{m+n-1}$, ideoque poſito

$x^n = \frac{f^n}{g^n}\cos n\phi$ ob $P = x^0$ & $\mathfrak{P} = 1$.

$$\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}}\cos(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\phi$$

& $\mathfrak{p} = 0$; atque

$$\mathfrak{q} = \frac{maf^{m-1}}{g^{m-1}}\sin(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\phi$$

Ergo

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{m^2a^2f^{2(m-1)}}{g^{2(m-1)}} + \frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}}\cos n\phi + \frac{(m+n)^2b^2f^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Quodſi

Quodſi vero eſt $ff - 2fgx\,\text{coſ}\,\phi + ggxx$ diviſor ipſius $a + bx^n$, erit $a + \frac{bf^n}{g^n}\,\text{coſ}\,n\phi = 0$ & $\frac{bf^n}{g^n}\,\text{ſin}\,n\phi = 0$, unde $aa = \frac{bbf^{2n}}{g^{2n}}$. Erit ergo:

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{(m+n)^2 bbf^{2(m+n-1)}}{g^{2(m+n-1)}} - \frac{m(2n+m)aaf^{2(m-1)}}{g^{2(m-1)}}$$

$$= \frac{nnaaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nnbbf^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Deinde vero erit:

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}}\,\text{ſin}\,(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\,\text{ſin}\,(m+n-1)\phi$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}}\,[(m+n)\,\text{ſin}\,(m+n-1)\phi - m\,\text{coſ}\,n\phi.\,\text{ſin}\,(m-1)\phi]$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}}\,[n\,\text{coſ}\,n\phi\,\text{ſin}\,(m-1)\phi + (m+n)\,\text{ſin}\,n\phi\,\text{coſ}\,(m-1)\phi]$$

$$\text{\&}\quad \mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} =$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}}\,[(m+n)\,\text{coſ}\,(m+n-1)\phi - m\,\text{coſ}\,n\phi.\,\text{coſ}\,(m-1)\phi]$$

Vel cum $ff - 2fg\,\text{coſ}\,\phi + ggxx$ ſit quoque diviſor ipſius $ax^{m-1} + bx^{m+n-1}$, erit:

$$\frac{af^{m-1}}{g^{m-1}}\,\text{coſ}\,(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}}\,\text{coſ}\,(m+n-1)\phi = 0$$

$$\text{\&}\quad \frac{af^{m-1}}{g^{m-1}}\,\text{ſin}\,(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}}\,\text{ſin}\,(m+n-1)\phi = 0,$$

unde erit:

$$\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\,\text{coſ}\,(m+n-1)\phi \;\text{\&}\; \mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\,\text{ſin}\,(m+n-1)\phi$$

ſeu

$$\mathfrak{Q} = \frac{-naf^{m-1}}{g^{m-1}}\,\text{coſ}\,(m-1)\phi \;\text{\&}\; \mathfrak{q} = \frac{-naf^{m-1}}{g^{m-1}}\,\text{ſin}\,(m-1)\phi.$$

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Ex quibus reſultabit fractio quaeſita:

$$+\frac{2g^m\,[f\cos m\,\phi - g\,x\cos(m-1)\,\phi\,]}{naf^{m-1}(ff - 2fgx\cos\phi + ggxx)}.$$

Quae formula ex priori exemplo ſequitur, ſi ponatur m negativum, unde non opus fuiſſet hunc caſum peculiarem conſtituiſſe.

EXEMPLUM III.

Si huius fractionis $\frac{x^m}{a + bx^n + cx^{2n}}$ *denominator habuerit factorem* $ff - 2fgx\cos\phi + ggxx$, *fractionem ſimplicem inveſtigare ex hoc factore oriundam.*

Si $ff - 2fgx\cos\phi + ggxx$ eſt factor denominatoris $a + bx^n + cx^{2n}$, erit ut ſupra oſtendimus:

$$a + \frac{bf^n}{g^n}\cos n\,\phi + \frac{cf^{2n}}{g^{2n}}\cos 2n\,\phi = 0$$

$$\&\ \frac{bf^n}{g^n}\sin n\,\phi + \frac{cf^{2n}}{g^{2n}}\sin 2n\,\phi = 0.$$

Cum igitur ſit $P = x^m$ & $Q = a + bx^n + cx^{2n}$, erit $\frac{dQ}{dx} = nbx^{n-1} + 2ncx^{2n-1}$; unde efficitur:

$$\mathfrak{P} = \frac{f^m}{g^m}\cos m\,\phi \quad \& \quad \mathfrak{p} = \frac{f^m}{g^m}\sin m\,\phi:$$

$$\mathfrak{Q} = \frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\,\phi + \frac{2ncf^{2n-1}}{g^{2n-1}}\cos(2n-1)\,\phi$$

$$\mathfrak{q} = \frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\,\phi + \frac{2ncf^{2n-1}}{g^{2n-1}}\sin(2n-1)\,\phi.$$

Quamobrem habebimus:

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2(n-1)}}{g^{2(n-1)}}\left(bb + \frac{4bcf^n}{g^n}\cos n\,\phi + \frac{4ccf^{2n}}{g^{2n}}\right)$$

At ex duabus prioribus aequationibus eſt:

$$\frac{f^{2n}}{g^{2n}}\left(bb + \frac{2bcf^n}{g^n}\cos n\,\phi + \frac{ccf^{2n}}{g^{2n}}\right) = aa;$$

ideo-

ideoque

$$\frac{4bcf^n}{g^n}\operatorname{cof} n\phi = \frac{2g^{2n}aa}{f^{2n}} - 2bb - \frac{2ccf^{2n}}{g^{2n}}$$

quo valore ibi ſubſtituto erit:

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2n-2}}{g^{2n-2}}\left(\frac{2aag^{2n}}{f^{2n}} - bb + \frac{2ccf^{2n}}{g^{2n}}\right)$$

ſeu

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2(2aag^{4n} - bbf^{2n}g^{2n} + 2ccf^{4n})}{ffg^{4n-2}}$$

Deinde erit: $\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} =$

$$\frac{nbf^{m+n-1}}{g^{m+n-1}}\operatorname{fin}(n-m-1)\phi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\operatorname{fin}(2n-m-1)\phi$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} =$$

$$\frac{nbf^{m+n-1}}{g^{m+n-1}}\operatorname{cof}(n-m-1)\phi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\operatorname{cof}(2n-m-1)\phi.$$

Quibus valoribus inventis erit fractio ſimplex quaeſita:

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\operatorname{fin}\phi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\operatorname{cof}\phi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\operatorname{cof}\phi + ggxx)}$$

417. Hae autem fractiones facilius exprimentur, ſi ipſos denominatorum factores determinemus. Sit igitur denominator fractionis propoſitae: $a + bx^n$ cuius factor trinomialis ſi ponatur: $ff - 2fgx\operatorname{cof}\phi + ggxx$ erit uti in Introductione oſtendimus:

$$a + \frac{bf^n}{g^n}\operatorname{cof} n\phi = 0 \quad \& \quad \frac{bf^n}{g^n}\operatorname{fin} n\phi = 0,$$

cum igitur ſit $\operatorname{fin} n\phi = 0$, erit vel $n\phi = (2k-1)\pi$, vel $n\phi = 2k\pi$, priori caſu erit $\operatorname{cof} n\phi = -1$, poſteriori $\operatorname{cof} n\phi = +1$. Si ergo a & b ſint quantitates affirmativae, prior caſus ſolus locum habebit, quo ſit $a = \frac{bf^n}{g^n}$; ac propterea: $f = a^{\frac{1}{n}}$ & $g = b^{\frac{1}{n}}$

reti-

retineamus autem loco harum quantitatum irrationalium litteras f & g, ſeu ponamus potius $a = f^n$ & $b = g^n$, ita ut factores inveſtigari debeant huius functionis $f^n + g^n x^n$. Cum igitur ſit $\phi = \frac{(2k-1)\pi}{n}$, ubi k numerum quemcunque affirmativum integrum deſignare poteſt; at vero maiores numeri pro k non ſunt ſumendi, quam qui reddant $\frac{2k-1}{n}$ unitate minorem; hinc functionis propoſitae $f^n + g^n x^n$ factores erunt ſequentes:

$$ff - 2fgx \operatorname{cof} \frac{\pi}{n} + ggxx$$

$$ff - 2fgx \operatorname{cof} \frac{3\pi}{n} + ggxx$$

$$ff - 2fgx \operatorname{cof} \frac{5\pi}{n} + ggxx$$

&c.

ubi notandum eſt ſi n ſit numerus impar, unum factorem haberi binomium hunc: $f + gx$; ſin autem n ſit numerus par nullus factor aderit binomius.

EXEMPLUM I.

Reſolvere hanc fractionem $\frac{x^m}{f^n + g^n x^n}$ *in ſuas fractiones ſimplices.*

Cum denominatoris unuſquiſque factor trinomialis contineatur in hac forma: $ff - 2fgx \operatorname{cof} \frac{(2k-1)\pi}{n} + ggxx$, erit in §. praecedente Exempl. I. $a = f^n$, $b = g^n$, & $\phi = \frac{(2k-1)\pi}{n}$, unde erit:

$\operatorname{ſin}(n-m-1)\phi = \operatorname{ſin}(m+1)\phi = \operatorname{ſin} \frac{(m+1)(2k-1)\pi}{n}$ &

cof

$$\cos(n-m-1)\varphi = -\cos(m+1)\varphi = -\cos\frac{(m+1)(2k-1)\pi}{n}.$$

Hinc ex iſto factore oritur fractio ſimplex haec:

$$\frac{2f\sin\frac{(2k-1)\pi}{n}\sin\frac{(m+1)(2k-1)\pi}{n} - 2\cos\frac{(m+1)(2k-1)\pi}{n}\left(gx - f\cos\frac{(2k-1)\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{(2k-1)\pi}{n} + ggxx\right)}.$$

Quamobrem fractio propoſita reſolvetur in has ſimplices:

$$\frac{2f\sin\frac{\pi}{n}.\sin\frac{(m+1)\pi}{n} - 2\cos\frac{(m+1)\pi}{n}\left(gx - f\cos\frac{\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{\pi}{n} - ggxx\right)}$$

$$\frac{2f\sin\frac{3\pi}{n}.\sin\frac{3(m+1)\pi}{n} - 2\cos\frac{3(m+1)\pi}{n}\left(gx - f\cos\frac{3\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{3\pi}{n} + ggxx\right)}$$

$$\frac{2f\sin\frac{5\pi}{n}.\sin\frac{5(m+1)\pi}{n} - 2\cos\frac{5(m+1)\pi}{n}\left(gx - f\cos\frac{5\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{5\pi}{n} + ggxx\right)}$$

&c.

Si ergo n fuerit numerus par, hoc modo omnes oriuntur fractiones ſimplices; ſin autem n ſit numerus impar, ob factorem binomium $f+gx$, ad fractiones hoc modo reſultantes inſuper addi debet haec:

$$\frac{\pm 1}{nf^{n-m-1}g^m(f+gx)}$$

ubi ſignum $+$ valet, ſi m fuerit numerus par, contra ſignum $-$. Si m fuerit numerus maior quam n, tum ad has fractiones accedent inſuper partes integrae huiuſmodi

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$$\mathrm{A}x^{m-n} + \mathrm{B}x^{m-2n} + \mathrm{C}x^{m-3n} + \mathrm{D}x^{m-4n} + \&c.$$

quamdiu exponentes manent affirmativi, eritque:

$$\mathrm{A}g^n = 1\,; \quad \text{Ergo} \quad \mathrm{A} = \frac{1}{g^n}$$

$$\mathrm{A}f^n + \mathrm{B}g^n = 0\ .\ .\ \mathrm{B} = -\frac{f^n}{g^{2n}}$$

$$\mathrm{B}f^n + \mathrm{C}g^n = 0\ .\ .\ \mathrm{C} = +\frac{f^{2n}}{g^{3n}}$$

$$\mathrm{C}f^n + \mathrm{D}g^n = 0\ .\ .\ \mathrm{D} = -\frac{f^{3n}}{g^{4n}}$$

&c. &c.

EXEMPLUM II.

Reſolvere hanc fractionem $\frac{1}{x^m(f^n + g^n x^n)}$ *in ſuas fractiones ſimplices.*

Quod ad factores ipſius $f + g^n x^n$ attinet, ex iis oriuntur eaedem fractiones, quas Exemplo praecedente eruimus, dummodo ibi ſumatur m negative: ſupereſt igitur tantum, ut fractiones ſimplices ex denominatoris altero factore x^m definiamus, quod hoc modo commodiſſime fit: ſtatuatur fractio propoſita $= \frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{N}x^{n-m}}{f^n + g^n x^n}$, eritque

$$\mathfrak{A}f^n = 1\,; \ \text{Ergo}\ \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A}g^n + \mathfrak{N} = 0\ .\ .\ \mathfrak{N} = -\frac{g^n}{f^n}.$$

Si $n-m$ adhuc fuerit numerus negativus, ſimili modo erit operandum, ita ut, ſi m fuerit numerus quantumvis magnus, reſultent huiuſmodi fractiones ſimplices

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \&c.$$

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cuius ſeriei tot termini ſunt ſumendi, quot habentur ipſius x exponentes affirmativi in denominatore. Eritque

$$\mathfrak{A}f^n = 1 \quad ; \qquad \text{Ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A}g^n + \mathfrak{B}f^n = 0 \quad . \quad . \quad . \quad \mathfrak{B} = -\frac{g^n}{f^{2n}}$$

$$\mathfrak{B}g^n + \mathfrak{C}f^n = 0 \quad . \quad . \quad . \quad \mathfrak{C} = +\frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{C}g^n + \mathfrak{D}f^n = 0 \quad . \quad . \quad . \quad \mathfrak{D} = -\frac{g^{3n}}{f^{4n}}$$

&c. &c.

Fractio ergo propoſita omnino in has fractiones ſimplices reſolvetur :

$$\frac{1}{f^n x^m} - \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} - \frac{g^{3n}}{f^{4n} x^{m-3n}} + \&c.$$

$$\frac{-2fg^m \sin\frac{\pi}{n} . \sin\frac{(m-1)\pi}{n} - 2g^m \cos\frac{(m-1)\pi}{n}\left(gx - f\cos\frac{\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{3\pi}{n} . \sin\frac{3(m-1)\pi}{n} - 2g^m \cos\frac{3(m-1)\pi}{n}\left(gx - f\cos\frac{3\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{3\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{5\pi}{n} . \sin\frac{5(m-1)\pi}{n} - 2g^m \cos\frac{5(m-1)\pi}{n}\left(gx - f\cos\frac{5\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{5\pi}{n} + ggxx\right)}$$

&c.

Quibus formulis ſi n fuerit numerus impar, ob $f + gx$ facto-

ctorem denominatoris, inſuper adiici debet:

$$\frac{\pm g^m}{nf^{n+m-1}(f+gx)}$$

ubi ſignorum ambiguorum $\pm$ ſuperius valet, ſi m fuerit numerus par, inferius vero ſi m impar.

418. Conſideremus nunc quoque formulam $a+bx^n$, ſi b ſit numerus negativus, ſitque propoſita haec functio:

$$f^n - g^n x^n$$

cuius primo ſemper erit factor $f-gx$; atque ſi n ſit numerus par, quoque $f+gx$ eius erit factor. Reliqui vero erunt trinomiales, quorum forma generalis ſi ponatur

$$ff - 2fgx\cos\varphi + ggxx$$

erit $f^n - f^n\cos n\varphi = 0$ & $f^n \sin n\varphi = 0$ ſive $\sin n\varphi = 0$ & $\cos n\varphi = 1$. Quibus ut ſatisfiat, oportet eſſe $n\varphi = 2k\pi$ exiſtente k numero quocunque integro, atque propterea erit $\varphi = \frac{2k\pi}{n}$. Factor ergo generalis erit:

$$ff - 2fgx\cos\frac{2k\pi}{n} + ggxx$$

ſumendo ergo pro $2k$ omnes numeros pares exponente n minores, prodibunt factores trinomiales omnes:

$$ff - 2fgx\cos\frac{2\pi}{n} + ggxx$$

$$ff - 2fgx\cos\frac{4\pi}{n} + ggxx$$

$$ff - 2fgx\cos\frac{6\pi}{n} + ggxx$$

&c.

EXEMPLUM I.

Reſolvere hanc fractionem $\frac{x^m}{f^n - g^n x^n}$ *in ſuas fractiones ſimplices.*

Quoniam denominatoris factor eſt $f-gx$, inde orietur fra-

fractio huiuſmodi $\frac{\mathfrak{A}}{f-gx}$, ad cuius numeratorem inveniendum, ponatur $x^m = P$ & $f^n - g^n x^n = Q$, erit $\frac{dQ}{dx} = -ng^n x^{n-1}$, fietque $\mathfrak{A} = \frac{-gx^m}{-ng^n x^{n-1}} = \frac{x^m}{ng^{n-1} x^{n-1}}$, poſito $x = \frac{f}{g}$. Ergo erit $\mathfrak{A} = \frac{1}{nf^{n-m-1} g^m}$, hincque fractio ſimplex ex factore $f - gx$ orta erit:

$$\frac{1}{nf^{n-m-1} g^m (f-gx)}.$$

Si n ſit numerus par, quia tum denominatoris factor quoque eſt $f + gx$, ponatur fractio ſimplex inde oriunda

$$= \frac{\mathfrak{A}}{f+gx},\ \text{erit}\ \mathfrak{A} = \frac{-gx^m}{ng^n x^{n-1}} = \frac{-x^m}{ng^{n-1} x^{n-1}},$$

poſito $x = \frac{-f}{g}$. Fiet ergo ob $n-1$ numerum imparem $g^{n-1} x^{n-1} = -f^{n-1}$: at erit $x^m = \frac{\pm f^m}{g^m}$, ubi ſignum ſuperius valet, ſi m fuerit numerus par, inferius ſi m ſit numerus impar. Quare cum ſit $\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1} g^m}$, erit fractio ſimplex ex factore $f + gx$ oriunda haec:

$$\frac{\mp 1}{nf^{n-m-1} g^m (f+gx)}.$$

Deinde cum factorum trinomialium forma generalis ſit:

$$ff - 2fgx \operatorname{coſ} \frac{2k\pi}{n} + ggxx,$$

ſi comparatio cum Exemplo I. §. 416. inſtituatur, erit $a = f^n$, $b = -g^n$ & $\varphi = \frac{2k\pi}{n}$; unde ſin $n\varphi = 0$ & coſ $n\varphi = 1$;

 at-

atque $\text{ſin}\,(n-m-1)\phi = -\text{ſin}\,(m+1)\phi = -\text{ſin}\,\frac{2k(m+1)\pi}{n}$,

& $\text{coſ}\,(n-m-1)\phi = \text{coſ}\,(m+1)\phi = \text{coſ}\,\frac{2k(m+1)\pi}{n}$.

Ex quibus erit fractio ſimplex hinc oriunda:

$$\frac{2f\,\text{ſin}\,\frac{2k\pi}{n}\,.\,\text{ſin}\,\frac{2k(m+1)\pi}{n} - 2\,\text{coſ}\,\frac{2k(m+1)\pi}{n}\left(gx - f\,\text{coſ}\,\frac{2k\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\,\text{coſ}\,\frac{2k\pi}{n} + ggxx\right)}$$

Hancobrem fractiones ſimplices quaeſitae erunt:

$$\frac{1}{nf^{n-m-1}g^m(f-gx)}$$

$$+\frac{2f\,\text{ſin}\,\frac{2\pi}{n}\,.\,\text{ſin}\,\frac{2(m+1)\pi}{n} - 2\,\text{coſ}\,\frac{2(m+1)\pi}{n}\left(gx - f\,\text{coſ}\,\frac{2\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\,\text{coſ}\,\frac{2\pi}{n} + ggxx\right)}$$

$$+\frac{2f\,\text{ſin}\,\frac{4\pi}{n}\,.\,\text{ſin}\,\frac{4(m+1)\pi}{n} - 2\,\text{coſ}\,\frac{4(m+1)\pi}{n}\left(gx - f\,\text{coſ}\,\frac{4\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\,\text{coſ}\,\frac{4\pi}{n} + ggxx\right)}$$

$$+\frac{2f\,\text{ſin}\,\frac{6\pi}{n}\,.\,\text{ſin}\,\frac{6(m+1)\pi}{n} - 2\,\text{coſ}\,\frac{6(m+1)\pi}{n}\left(gx - f\,\text{coſ}\,\frac{6\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\,\text{coſ}\,\frac{6\pi}{n} + ggxx\right)}$$

&c.

quibus ſi n fuerit numerus par, inſuper addi debet haec fractio: $\frac{\mp 1}{nf^{n-m-1}g^m(f+gx)}$

cu-

cuius ſignum ſuperius — eſt ſumendum, ſi m fuerit numerus par, inferius ſi impar. Praeterea vero ſi m ſit numerus non minor quam n, adiiciendae ſunt partes integrae:

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \&c.$$

quamdiu exponentes non fuerint negativi, eritque:

$$-Ag^n = 1 \quad \text{ſeu} \quad A = -\frac{1}{g^n}$$

$$Af^n - Bg^n = 0 \quad . \; . \quad B = -\frac{f^n}{g^{2n}}$$

$$Bf^n - Cg^n = 0 \quad . \; . \quad C = -\frac{f^{2n}}{g^{3n}}$$

$$Cf^n - Dg^n = 0 \quad . \; . \quad D = -\frac{f^{3n}}{g^{4n}}$$

&c. &c.

EXEMPLUM II.

Reſolvere hanc fractionem $\frac{1}{x^m(f^n - g^n x^n)}$ *in ſuas fractiones ſimplices.*

Fractiones quae ex denominatoris factore $f^n - g^n x^n$ oriuntur, eaedem erunt quae ante, dummodo in illis formulis m negative accipiatur. Quare ad alterum factorem x^m eſt reſpiciendum, ex quo ſi ponamus has fractiones reſultare:

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \&c.$$

quae ſeries eouſque eſt continuanda, donec exponentes ipſius x fiant negativi. Erit vero

$$\mathfrak{A}f^n = 1 \quad ; \quad \text{Ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{B}f^n - \mathfrak{A}g^n = 0 \quad . \; . \quad \mathfrak{B} = \frac{g^n}{f^{2n}}$$

$$\mathfrak{C}f^n - \mathfrak{B}g^n = 0 \quad . \; . \quad \mathfrak{C} = \frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{D}f^n - \mathfrak{C}g^n = 0 \quad . \; . \quad \mathfrak{D} = \frac{g^{3n}}{f^{4n}}$$

&c. &c.

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Fractio ergo propoſita reſolvetur in has fractiones ſimplices:

$$\frac{1}{f^n x^m} + \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} + \frac{g^{3n}}{f^{4n} x^{m-3n}} + \&c.$$

$$+ \frac{g^m}{n f^{n+m-1}(f - gx)}$$

$$\frac{-2fg^m \sin\frac{2\pi}{n}\sin\frac{2(m-1)\pi}{n} - 2g^m \cos\frac{2(m-1)\pi}{n}\left(gx - f\cos\frac{2\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{2\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{4\pi}{n}\sin\frac{4(m-1)\pi}{n} - 2g^m \cos\frac{4(m-1)\pi}{n}\left(gx - f\cos\frac{4\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{4\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{6\pi}{n}\sin\frac{6(m-1)\pi}{n} - 2g^m \cos\frac{6(m-1)\pi}{n}\left(gx - f\cos\frac{6\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{6\pi}{n} + ggxx\right)}$$

&c.

quibus ſi n fuerit numerus par, inſuper addi debet haec fractio:

$$\frac{\mp g^m}{nf^{n+m-1}(f + gx)}$$

quae autem praetermittitur, ſi n fuerit numerus impar. Signorum ambiguorum vero ſuperius — valet, ſi m ſit numerus par, inferius vero +, ſi m ſit numerus impar.

419. Hoc ergo modo omnes fractiones, quarum denominator ex duobus conſtat membris huiuſmodi $a + bx^n$, in fractiones ſimplices reſolvuntur. At ſi denominator conſtet tribus huiuſmodi membris $a + bx^n + cx^{2n}$, tum primum videndum eſt, utrum is in duos factores reales prioris formae reſolvi poſſit. Hoc enim ſi eveniat, reſolutio in fractiones ſimpli-

plices modo ante expoſito inſtitui poterit. Si enim proponatur huiuſmodi fractio

$$\frac{x^m}{(f^n+g^nx^n)(f^n+h^nx^n)}$$

ea primum in duas fractiones transformabitur huiuſmodi:

$$\frac{\alpha x^m}{f^n+g^nx^n}+\frac{\beta x^m}{f^n+h^nx^n}$$

eritque $\alpha f^n+\beta f^n=1$ & $\alpha h^n+\beta g^n=0$, unde fit

$\alpha=\frac{1}{f^n}-\beta=-\frac{\beta g^n}{h^n}$; ideoque habebitur

$$\beta=\frac{h^n}{f^n(h^n-g^n)} \text{ \& } \alpha=\frac{g^n}{f^n(g^n-h^n)}.$$

Si exponens m fuerit maior quam n, tranſmutatio in ſequentes fractiones erit commodior:

$$\frac{\alpha x^{m-n}}{f^n+g^nx^n}+\frac{\beta x^{m-n}}{f^n+h^nx^n}$$

qua fit $\alpha+\beta=0$ & $\alpha h^n+\beta g^n=1$, ideoque $\alpha=\frac{1}{h^n-g^n}$ & $\beta=\frac{1}{g^n-h^n}$. Utra autem transformatio adhibeatur, utraque fractio hoc modo oriunda methodo ante expoſita reſolvetur in ſuas fractiones ſimplices, quae iunctim ſumtae fractioni propoſitae erunt aequales.

420. Simili modo methodus hactenus tradita ſufficiet, ſi denominator ex pluribus membris conſtet huiuſmodi $a+bx^n+cx^{2n}+dx^{3n}+ex^{4n}+$ &c. dummodo is in factores formae $f^n\pm g^nx^n$ reſolvi queat. Ponamus enim occurrere hanc fractionem in ſuas fractiones ſimplices reſolvendam:

 x^m

$$\frac{x^m}{(a-x^n)(b-x^n)(c-x^n)(d-x^n)}.$$

Haec primum reſolvetur in has:

$$\frac{Ax^m}{a-x^n}+\frac{Bx^m}{b-x^n}+\frac{Cx^m}{c-x^n}+\frac{Dx^m}{d-x^n}$$

quarum numeratores ſequenti modo determinabuntur, ut ſit

$$A=\frac{1}{(b-a)(c-a)(d-a)}$$

$$B=\frac{1}{(a-b)(c-b)(d-b)}$$

$$C=\frac{1}{(a-c)(b-c)(d-c)}\ \&c.$$

Hac ergo praeparatione facta, ſingulae iſtae fractiones methodo ante expoſita in ſuas fractiones ſimplices reſolventur; quae cunctae in unam ſummam erunt colligendae.

421. Quodſi vero huiuſmodi denominator $a+bx^n+cx^{2n}+dx^{3n}+$ &c. non omnes factores formae $f^n+g^nx^n$ habeant reales, bini imaginarii erunt coniungendi. Ponamus ergo huiuſmodi binorum factorum productum eſſe:

$$f^{2n}-2f^ng^nx^n\operatorname{coſ}\omega+g^{2n}x^{2n}.$$

& cum haec expreſſio nullos habeat factores ſimplices reales, ponamus factores trinomiales in hac forma generali contineri:

$$ff-2fgx\operatorname{coſ}\phi+ggxx$$

quorum numerus erit $=n$. Poſito ergo $x^n=\frac{f^n}{g^n}\operatorname{coſ}n\phi$; orietur haec aequatio:

$$1-2\operatorname{coſ}\omega.\operatorname{coſ}n\phi+\operatorname{coſ}2n\phi=0.$$

Deinde poſito $x^n=\frac{f^n}{g^n}\operatorname{ſin}n\phi$; erit quoque:

$$-2\operatorname{coſ}\omega.\operatorname{ſin}n\phi+\operatorname{ſin}2n\phi=0$$

quae diviſa per $\operatorname{ſin}n\phi$ dat $\operatorname{coſ}n\phi=\operatorname{coſ}\omega$, ſicque ſimul priori

ri aequationi ſatisfit. Erit ergo $n\phi = 2k\pi \pm \omega$ denotante k numerum quemvis integrum, ideoque erit $\phi = \frac{2k\pi \pm \omega}{n}$, & factores omnes continebuntur in hac forma:

$ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx$ unde ſequentes habebuntur factores:

$$ff - 2fgx \cos \frac{\omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi + \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi + \omega}{n} + ggxx \quad \&c.$$

quorum tot ſunt ſumendi, donec eorum numerus fiat $= n$.

422. Si igitur proponatur iſta fractio in ſuas fractiones ſimplices reſolvenda:

$$\frac{x^{m-1}}{f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n}}$$

quoniam denominatoris factor trinomialis quicunque continetur in hac forma: $ff - 2fgx \cos \phi + ggxx$ exiſtente $\phi = \frac{2k\pi \pm \omega}{n}$, conſideretur iſta fractio:

$$\frac{x^m}{f^{2n}x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1}}$$

illi aequalis, ac ponatur numerator $x^m = P$ ac denominator $f^{2n}x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1} = Q$: erit

$\frac{dQ}{dx} = f^{2n} - 2(n+1) f^n g^n x^n \cos\omega + (2n+1) g^{2n} x^{2n}$.

Hinc ponendo $x^n = \frac{f^n}{g^n}\cos n\phi$; erit: $\mathfrak{P} = \frac{f^m}{g^m}\cos m\phi$

ſeu $\mathfrak{P} = \frac{f^m}{g^m}\cos\frac{m(2k\pi\pm\omega)}{n}$ &

$\mathfrak{Q} = f^{2n}[1-2(n+1)\cos\omega\cos n\phi + (2n+1)\cos 2n\phi]$. Cum autem ſit $\cos n\phi = \cos\omega$, erit $\cos 2n\phi = 2\cos\omega^2 - 1$; ideoque $\mathfrak{Q} = f^{2n}(-2n + 2n\cos\omega^2) = -2nf^{2n}\sin\omega^2$.

Deinde poſito $x^n = \frac{f^n}{g^n}\sin n\phi$, fiet:

$$\mathfrak{p} = \frac{f^m}{g^m}\sin m\phi = \frac{f^m}{g^m}\sin\frac{m(2k\pi\pm\omega)}{n} \quad \&$$

$$\mathfrak{q} = -f^{2n}[2(n+1)\cos\omega\sin n\phi - (2n+1)\sin 2n\phi]$$

ob $\sin 2n\phi = 2\sin n\phi\cos n\phi = 2\cos\omega\sin n\phi$; erit $\mathfrak{q} = 2nf^{2n}\cos\omega\sin n\phi$. Cum autem ſit $n\phi = 2k\pi\pm\omega$, erit $\sin n\phi = \pm\sin\omega$ & $\mathfrak{q} = \pm 2nf^{2n}\sin\omega\cdot\cos\omega$. His inventis erit: $\mathfrak{Q}^2+\mathfrak{q}^2 = 4n^2f^{4n}\sin\omega^2$

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{2nf^{m+2n}}{g^m}(\pm\cos m\phi.\sin\omega.\cos\omega + \sin m\phi.\sin\omega^2)$$

ſive $\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\cos(m\phi\mp\omega)$ ſeu

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\cos\frac{2km\pi\pm(m-n)\omega}{n}$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{2nf^{m+2n}}{g^m}(-\cos m\phi\sin\omega^2 \pm \sin m\phi.\sin\omega\cos\omega)$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\sin(m\phi\mp\omega) \quad \text{ſeu}$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\sin\frac{2km\pi\pm(m-n)\omega}{n}.$$

Hinc

Hinc ex denominatoris factore: $ff-2fgx\cos\frac{2k\pi\pm\omega}{n}+ggxx$

naſcitur iſta fractio ſimplex:

$$\frac{\pm f\sin\frac{2k\pi\pm\omega}{n}\cos\frac{2km\pi\pm(m-n)\omega}{n}\pm\sin\frac{2km\pi\pm(m-n)\omega}{n}\left(gx-f\cos\frac{2k\pi\pm\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2k\pi\pm\omega}{n}+ggxx\right)}$$

ſeu

$$\frac{\pm gx\sin\frac{2km\pi\pm(m-n)\omega}{n}\mp f\sin\frac{2k(m-1)\pi\pm(m-n-1)\omega}{n}}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2k\pi\pm\omega}{n}+ggxx\right)}$$

EXEMPLUM.

Reſolvere hanc fractionem $\frac{x^{m-1}}{f^{2n}-2f^ng^nx^n\cos\omega+g^{2n}x^{2n}}$ *in ſuas fractiones ſimplices.*

Iſtae fractiones ſimplices quaeſitae ergo erunt:

$$\frac{+f\sin\frac{\omega}{n}\cos\frac{(m-n)\omega}{n}+\sin\frac{(m-n)\omega}{n}\left(gx-f\cos\frac{\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{\omega}{n}+ggxx\right)}$$

$$\frac{-f\sin\frac{2\pi-\omega}{n}\cos\frac{2m\pi-(m-n)\omega}{n}-\sin\frac{2m\pi-(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi-\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi-\omega}{n}+ggxx\right)}$$

$$\frac{+f\sin\frac{2\pi+\omega}{n}\cos\frac{2m\pi+(m-n)\omega}{n}+\sin\frac{2m\pi+(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi+\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi+\omega}{n}+ggxx\right)}$$

$$-f\,\text{ſin}\,\frac{4\pi-\omega}{n}\,\text{coſ}\,\frac{4m\pi-(m-n)\omega}{n}-\text{ſin}\,\frac{4m\pi-(m-n)\omega}{n}\left(gx-f\,\text{coſ}\,\frac{4\pi-\omega}{n}\right)$$
$$\overline{nf^{2n-m}g^{m-1}\,\text{ſin}\,\omega\left(ff-2fgx\,\text{coſ}\,\frac{4\pi-\omega}{n}+ggxx\right)}$$
$$+f\,\text{ſin}\,\frac{4\pi+\omega}{n}\,\text{coſ}\,\frac{4m\pi+(m-n)\omega}{n}+\text{ſin}\,\frac{4m\pi+(m-n)\omega}{n}\left(gx-f\,\text{coſ}\,\frac{4\pi+\omega}{n}\right)$$
$$\overline{nf^{2n-m}g^{m-1}\,\text{ſin}\,\omega\left(ff-2fgx\,\text{coſ}\,\frac{4\pi+\omega}{n}+ggxx\right)}$$

&c.

ſicque eouſque erit progrediendum, quoad harum fractionum numerus fuerit n. Si m fuerit numerus vel maior quam $2n-1$ vel numerus negativus, priori caſu partes integrae, poſteriori vero fractiones inſuper ſunt adiiciendae, quae modo ante expoſito facile inveniuntur.

FINIS.

SUPPLEMENTUM

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EDITORIS MONITUM.

AD exitum iam properabat huius Operis impreſſio, cum Celeberrimus Io. ALBERTUS EULERUS Auctoris noſtri filius, & Petropolitanae Academiae a ſecretis, datis ad Cl. Matheſeos Sublimioris Profeſſorem GREGORIUM FONTANAM litteris, a quo fuerat hac de re officioſe rogatus, magni Parentis ſui Diſſertationem ineditam, eamque oppido eximiam & numeris omnibus abſolutam ad nos miſit, cuius titulus eſt: *Dilucidationes in Capita poſtrema Calculi mei Differentialis de Functionibus Inexplicabilibus.*

C' eſt avec bien de plaiſir (inquit Cl. Io. ALBERTUS in ea epiſtola Petropoli die 18 Decembris anni proxime elapſi 1787 exarata) , *que je vous envoie la copie ci-jointe du Mémoire de ſeu mon Père ſur les Fonctions Inexplicables*, *que votre Ami & Elève ſouhaite de faire entrer dans la nouvelle édition qu' il va publier du Calcul Différentiel. Je vous l' aurois expédié plus tôt ſans la grande difficulté &c....* Haec autem una eſt ex centum octoginta tribus Diſſertationibus poſthumis, quas immortalis LEONARDUS Petropolitanae Academiae moriens reliquit, & in eius Commentariis deinceps edendas legavit.

Nihil porro nobis optabilius contingere poterat, quam ut nova haec Euleriani Operis editio tam opportuno, tamque

que excellenti additamento locupletaretur, & hoc ſaltem nomine Berolinenſi praeſtaret. Itaque Diſſertationem ipſam continuo hic imprimi curavimus, ut Adnotationes poſt eandem ſequentes, quaſi poſt lautiſſimas epulas vilius obſonium, indulgentius haberentur. Noſtram interim de te benemerendi voluntatem Lector benevole, aequi bonique conſule, fave conatibus noſtris, & vale.

DI-

DILUCIDATIONES

IN CAPITA POSTREMA CALCULI MEI DIFFERENTIALIS

DE FUNCTIONIBUS INEXPLICABILIBUS

1.

Cum hoc argumentum, utpote in Analyſi prorſus novum, neutiquam ſatis dilucide ſit pertractatum, conſtitui hic, idem maiori ſtudio retractare, atque omnia momenta, quibus innititur, ex primis principiis repetere; ubi plurimum iuvabit idonea ſigna in calculum introduxiſſe. Ita ſi propoſita fuerit ſeries quaecunque, eius terminos indicibus 1, 2, 3, 4, &c. reſpondentes his ſignis repraeſentabo (1), (2), (3), (4), &c. hincque ergo terminus generalis huius ſeriei, indici indefinito x reſpondens, mihi erit (x), qui ergo pro quavis ſerie certa erit functio ipſius x, quam penitus cognitam aſſumo, ita ſcilicet comparatam, ut eius valores non ſolum pro numeris integris, loco x aſſumtis, ſed etiam pro fractis, atque adeo ſurdis exhiberi queant.

2. Denotet porro $\Sigma : x$ terminum ſummatorium eiuſdem ſeriei, qui exprimat ſummam terminorum a primo incipientium uſque ad terminum (x), ita ut ſit

$$\Sigma : x = (1) + (2) + (3) + (4) + \cdots\cdots + (x),$$

cuius ergo omnes valores, quoties x fuerit numerus integer poſitivus ex ipſa ſerie actu exhiberi poterunt; ſiquidem erit ut ſequitur:

$$\Sigma : 1 = (1)$$
$$\Sigma : 2 = (1) + (2)$$
$$\Sigma : 3 = (1) + (2) + (3)$$
$$\Sigma : 4 = (1) + (2) + (3) + (4)$$
&c.

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Cuiuſmodi autem valores eadem formula $\Sigma : x$ ſit acceptura quando loco x numeri fracti, vel adeo ſurdi, ſive poſitivi, ſive negativi tribuantur, hinc neutiquam apparet, unde iſtos valores ad peculiare functionum genus, quas *inexplicabiles* vocavi, refero. Quemadmodum igitur tales functiones per formulas analyticas determinatas exprimi queant, hic imprimis ſum inveſtigaturus

3. Totum autem hoc negotium commodiſſime expedietur per differentias continuas ex ſerie propoſita derivatas, dum ſcilicet quilibet terminus a ſequente ſubtrahitur, quo pacto oritur ſeries primarum differentiarum, ex qua porro ſimili modo differentiae ſecundae, tertiae, quartae &c. formabuntur. Omnes autem has differentias ſequentibus characteribus indicabo

DIFFERENTIAE I.	DIFFERENTIAE II.	DIFFERENTIAE III.
$(2)-(1)=\Delta 1$	$\Delta 2-\Delta 1=\Delta^2 1$	$\Delta^2 2-\Delta^2 1=\Delta^3 1$
$(3)-(2)=\Delta 2$	$\Delta 3-\Delta 2=\Delta^2 2$	$\Delta^2 3-\Delta^2 2=\Delta^3 2$
$(4)-(3)=\Delta 3$	$\Delta 4-\Delta 3=\Delta^2 3$	$\Delta^2 4-\Delta^2 3=\Delta^3 3$
$(5)-(4)=\Delta 4$	$\Delta 5-\Delta 4=\Delta^2 4$	$\Delta^2 5-\Delta^2 4=\Delta^3 4$
&c.	&c.	&c.

4. His characteribus conſtitutis ſinguli ſeriei termini ex primo (1), eiuſque differentiis $\Delta 1$, $\Delta^2 1$, $\Delta^3 1$, $\Delta^4 1$, &c. exprimi poterunt. Cum enim ſit $(2)=(1)+\Delta 1$, & $\Delta 2=\Delta 1+\Delta^2 1$, ob $(3)=(2)+\Delta 2$, erit $(3)=(1)+2\Delta 1+\Delta^2 1$; hinc iam fluit iſta aequalitas $\Delta 3=\Delta 1+2\Delta^2 1+\Delta^3 1$. Quia nunc $(4)=(3)+\Delta 3$ habebimus $(4)=(1)+3\Delta 1+3\Delta^2 1+\Delta^3 1$. Inde porro ſequitur: $\Delta 4=\Delta 1+3\Delta^2 1+3\Delta^3 1+\Delta^4 1$, ob $(5)=(4)+\Delta 4$, erit $(5)=(1)+4\Delta 1+6\Delta^2 1+4\Delta^3 1+\Delta^4 1$, & ita porro. Ex ipſa formatione harum formularum manifeſtum eſt hic eoſdem coefficientes, qui in poteſtate binomiali habentur, cuius exponens eſt unitate minor quam index termini propoſiti, occurrere. Ita erit

$$(n)=(1)+\frac{n-1}{1}\Delta 1+\frac{n-1}{1}\cdot\frac{n-2}{2}\Delta^2 1+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\Delta^3 1+\&c.$$

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5. Quod ſi hic numerum n unitate augeamus habebimus:

$$(n+1)=(1)+\frac{n}{1}\Delta 1+\frac{n}{1}.\frac{n-1}{2}\Delta^2 1+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 1+\&c.$$

Cum iam haec poſtrema expreſſio exprimat terminum qui a primo n gradibus eſt remotus, ſimili modo terminus, qui a ſecundo per totidem gradus eſt promotus $(n+2)$ ex ſecundo eiuſque differentiis determinatur; Erit enim:

$$(n+2)=(2)+\frac{n}{1}\Delta 2+\frac{n}{1}.\frac{n-1}{2}\Delta^2 2+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 2+\&c.$$

Eodem modo evidens eſt fore protinus

$$(n+3)=(3)+\frac{n}{1}\Delta 3+\frac{n}{1}.\frac{n-1}{2}\Delta^2 3+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 3+\&c.$$

$$(n+4)=(4)+\frac{n}{1}\Delta 4+\frac{n}{1}.\frac{n-1}{2}\Delta^2 4+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 4+\&c.$$

6. Hinc ergo patet, ipſum ſeriei noſtrae terminum generalem (x) ex primo eiuſque differentiis hoc modo definiri:

$$(x)=(1)+\frac{x-1}{1}\Delta 1+\frac{x-1}{1}.\frac{x-2}{2}\Delta^2 1+\frac{x-1}{1}.\frac{x-2}{2}.\frac{x-3}{3}\Delta^3 1+\&c.$$

Hinc terminus ultimum ſequens $(x+1)$ erit:

$$(x+1)=(1)+\frac{x}{1}\Delta 1+\frac{x}{1}.\frac{x-1}{2}\Delta^2 1+\frac{x}{1}.\frac{x-1}{2}.\frac{x-2}{3}\Delta^3 1+\&c.$$

quae expreſſio, cum in ſequentibus frequentiſſime occurrat, brevitatis gratia introducamus ſequentes characteres

$$\frac{x}{1}=x$$

$$\frac{x}{1}.\frac{x-1}{2}=x'$$

$$\frac{x}{1}.\frac{x-1}{2}.\frac{x-2}{3}=x''$$

$$\frac{x}{1}.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}=x'''$$

&c.

 qui-

quibus adhibitis habebimus ſequentes aequationes:

$$(x+1)=(1)+x\Delta 1+x'\Delta^2 1+x''\Delta^3 1+\&c.$$
$$(x+2)=(2)+x\Delta 2+x'\Delta^2 2+x''\Delta^3 2+\&c.$$
$$(x+3)=(3)+x\Delta 3+x'\Delta^2 3+x''\Delta^3 3+\&c.$$
$$(x+4)=(4)+x\Delta 4+x'\Delta^2 4+x''\Delta^3 4+\&c.$$

. . . .

. . . .

$$(x+n)=(n)+x\Delta n+x'\Delta^2 n+x''\Delta^3 n+\&c.$$

7. Deinde etiam ſummas quotcunque terminorum noſtrae ſeriei ex ſolo termino primo eiuſque differentiis determinari poterit, quemadmodum ſequens tabula declarat:

$$\Sigma{:}1=(1)$$
$$\text{add. }(2)=(1)+\Delta 1$$
$$\Sigma{:}2=2(1)+\Delta 1$$
$$(3)=(1)+2\Delta 1+\Delta^2 1$$
$$\Sigma{:}3=3(1)+3\Delta 1+\Delta^2 1$$
$$(4)=(1)+3\Delta 1+3\Delta^2 1+\Delta^3 1$$
$$\Sigma{:}4=4(1)+6\Delta 1+4\Delta^2 1+\Delta^3 1$$
$$(5)=(1)+4\Delta 1+6\Delta^2 1+4\Delta^3 1+\Delta^4 1$$
$$\Sigma{:}5=5(1)+10\Delta 1+10\Delta^2 1+5\Delta^3 1+\Delta^4 1$$

&c.

Hic iterum evidens eſt coefficientes eoſdem eſſe, qui in poteſtate binomiali eiuſdem ordinis occurrunt

8. In uſum igitur vocatis characteribus modo ante ſtabilitis ipſum terminum ſummatorium noſtrae ſeriei $\Sigma{:}x$ exprimere valemus: Erit enim

$$\Sigma{:}x=x(1)+x'\Delta 1+x''\Delta^2 1+x'''\Delta^3 1+\&c.$$

quae forma iam ita eſt comparata, ut loco x non ſolum numeros integros, ſed etiam fractos, imo ſurdos, quoſcunque tam poſitivos quam negativos accipere liceat, quibus caſibus utique iſta expreſſio in infinitum progredietur, niſi forte ſeries propoſita deducat tandem ad differentias evaneſcentes, cuiuſmodi ſeries algebraicae vocari ſolent, quibus ergo caſibus non

non ad functiones inexplicabiles pervenitur. Interim tamen iſta expreſſio pro termino ſummatorio inventa quando in infinitum porrigitur nihil adiumenti affert, quando differentiationes, vel etiam ſummationes ſunt inſtituendae; quamobrem in id erit incumbendum, quemadmodum, ſaltem pro certis caſibus terminus ſummatorius inventus in alias formas transfundi queat, quae neque differentiationi neque integrationi refragentur, atque huc pertinent omnia ſubſidia, quae in calculo differentiali fuſius expoſui, & quorum inventio non parum erat obſtruſa: ſequenti autem modo totum hoc negotium facile conficietur.

9. Ad expreſſionem pro termino ſummatorio $\Sigma : x$ modo ante inventam addantur plures formulae ſub hac ſpecie contentae.

$$(n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + \&c. \quad \ldots \quad -(x+n)$$

quarum ſummae cum ſint nihilo aequales, omnes, quotcunque fuerint, cum $\Sigma : x$ iunctim ſumtae nihilominus terminum ſummatorium exprimenti. Sumantur ergo pro n ſucceſſive omnes numeri 1, 2, 3, 4, &c., & tota expreſſio ſecundum columnas verticales ſingulis valoribus x, x', x'', &c. reſpondentes ſequenti modo diſponantur:

Expressio generalis pro termino summatorio

$$
\begin{array}{l}
\quad x(1) + x'\Delta 1 + x''\Delta^2 1 + x'''\Delta^3 1 + \&c. \\
(1) + x\Delta 1 + x'\Delta^2 1 + x''\Delta^3 1 + x'''\Delta^4 1 + \&c. \ldots - (x+1) \\
(2) + x\Delta 2 + x'\Delta^2 2 + x''\Delta^3 2 + x'''\Delta^4 2 + \&c. \ldots - (x+2) \\
(3) + x\Delta 3 + x'\Delta^2 3 + x''\Delta^3 3 + x'''\Delta^4 3 + \&c. \ldots - (x+3) \\
\quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
(n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + x'''\Delta^4 n + \&c. \ldots - (x+n)
\end{array}
$$

10. Etſi veritas huius expreſſionis nulli amplius dubio eſt obnoxia, tamen non parum iuvabit, eam ex ipſa forma con-

confirmaſſe. Colligantur nimirum in unam ſummam ſingulae columnae verticales, ac prima quidem ſumma erit

$(1)+(2)+(3)+(4)+ \dots\dots + (n) = \Sigma : n$

ſecunda columna dat

$x[(1)+\Delta 1+\Delta 2+\Delta 3+ \dots\dots + \Delta n]$

Cum autem ſit

$\Delta 1=(2)-(1);\ \Delta 2=(3)-(2);\ \Delta 3=(4)-(3);$ &c.

tota haec ſumma contrahetur in $x(n+1)$.

Simili modo tertiae columnae ſumma erit

$x'[\Delta 1+\Delta^2 1+\Delta^2 2+\Delta^2 3+\Delta^2 4+ \dots\dots +\Delta^2 n]$

& quia

$\Delta^2 1=\Delta 2-\Delta 1;\ \Delta^2 2=\Delta 3-\Delta 2 \dots \Delta^2 n=\Delta(n+1)-\Delta n$

illa ſumma contrahitur in $x'\Delta(n+1)$. Eodem modo patet fore quartae columnae ſummam $x''\Delta^2(n+1)$, & quintae $= x'''\Delta^3(n+1)$, & ita porro. Ultimae vero columnae ſubtrahendae ſumma eſt

$(x+1)+(x+2)+(x+3)+ \dots\dots +(x+n)=\Sigma:(x+n)-\Sigma:x.$

11. Summa igitur omnium columnarum verticalium mediarum praeter primam & ultimam eſt ut vidimus

$x(n+1)+x'\Delta(n+1)+x''\Delta^2(n+1)+x'''\Delta^3(n+1)+$ &c.

Cum autem ſit $x(1)+x'\Delta 1+x''\Delta^2 1+x'''\Delta^3 1+$ &c. $=\Sigma:x$, ſingulis terminis numero n auctis erit ſumma noſtrae ſeriei $x(n+1)+x'\Delta(n+1)+x''\Delta^2(n+1)+$ &c. $=\Sigma:(x+n)-\Sigma:n$ conſequenter omnium plane columnarum ſumma praeter ultimam eſt $=\Sigma:(x+n)$: unde ſi ſumma ultimae columnae, quae eſt $\Sigma:(x+n)-\Sigma:x$ ſubtrahatur, remanebit ſumma totius figurae $=\Sigma:x$, hoc eſt terminus ſummatorius quaeſitus.

12. Maxime hic mirum videbitur, quod valorem formulae $\Sigma:x$, quae ſerie ſatis ſimplici exprimitur, per congeriem innumerabilium ſerierum expreſſum & involutum dederimus; verum mox ſummus uſus huius formae complicatiſſimae patebit, quando numerum ſerierum horizontalium adeo in infinitum continuaverimus, quod fiet, ſi pro n numerum infinitum accipiamus, quemadmodum nunc clarius explicabimus.

13.

13. Denotante igitur n numerum infinite magnum, ſumma ſecundae columnae verticalis, quae eſt $x(n+1)$ continebit terminum ſeriei noſtrae infiniteſimum, qui ergo ſi evaneſcat, multo magis ſummae ſequentium columnarum verticalium evaneſcent; quamobrem hoc caſu ſufficiet ſolam primam columnam cum ultima in calculo retinuiſſe. Sin autem termini infiniteſimi non evaneſcant, ſed tamen inter ſe fuerint aequales, tum tertiam columnam cum ſequentibus abiicere licebit. Porro autem ſi demum differentiae ſecundae infiniteſimae evaneſcant, tres priores columnas verticales in calculo retineri debebunt, ſimilique modo quatuor ſi tertiae demum infiniteſimae evaneſcant. Secundum igitur hoc ſerierum diſcrimen, ipſas ſeries in ſequentes ſpecies diſtribuemus:

Species prima ſerierum,

quarum termini infiniteſimi evaneſcunt

14. Quoties igitur tales ſeries proponuntur, pro earum termino ſummatorio ſufficiet terminos primae & ultimae columnae verticalis in calculo retinuiſſe, ſicque nanciſcemur pro termino ſummatorio ſequentem expreſſionem

$$\Sigma : x = \begin{array}{l} (1) + (2) + (3) + (4) + \&c. \\ -(x+1)-(x+2)-(x+3)-(x+4)-\&c. \end{array}$$

quae quidem in infinitum excurrit, atque eo magis converget quo minor fuerit index x, quandoquidem ſi evaneſcat, tota ſeries in nihilum abibit, ſive erit $\Sigma : 0 = 0$, id quod cum rei natura egregie congruit; quando enim numerus terminorum addendorum eſt nullus, etiam ſumma neceſſario debet eſſe nulla.

15. Quando autem index x numerus eſt praemagnus, haec ſeries utique parum converget, verum ſemper licebit huiuſmodi caſus ad indices minores reducere. Cum enim ſit $\Sigma : (x+1) = \Sigma : x + (x+1)$, ſimili erit modo

Σ:

$\Sigma:(x+2)=\Sigma:x+(x+1)+(x+2)$, atque adeo in genere, denotante i numerum integrum

$\Sigma:(x+i)=\Sigma:x+(x+1)+(x+2)+\ldots+(x+i)$ quamobrem ſi ſumma $x+i$ terminorum deſideretur, ſufficiet ſummam x terminorum, hoc eſt $\Sigma:x$ inveſtigaſſe, hocque modo omnes huiuſmodi quaeſtiones reduci poterunt ad caſus, ubi index x eſt adeo unitate minor, quo caſu ſeries pro $\Sigma:x$ ante data vehementer converget.

16. Talis reductio imprimis eſt neceſſaria, quando index x eſt numerus negativus. Cum enim ſit $\Sigma:x=\Sigma:(x-1)+(x)$, erit $\Sigma:(x-1)=\Sigma:x-(x)$, eodemque modo $\Sigma:(x-2)=\Sigma:x-(x)-(x-1)$, & $\Sigma:(x-3)=\Sigma:x-(x)-(x-1)-(x-2)$ & in genere $\Sigma:(x-i)=\Sigma:x-(x)-(x-1)\ldots-(x-i+1)$ hocque modo quantumvis numerus negativus $x-i$ fuerit magnus, reſolutio ſemper ad $\Sigma:x$ reduci poteſt, ita ut ſit $x<1$.

EXEMPLUM.

17. Propoſita ſit haec ſeries

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots\frac{1}{x}=\Sigma:x$$

ita ut huius ſeriei harmonicae ſumma x terminorum deſideretur, ubi pro x numeros quoſcunque praeter integros poſitivos accipere licet, ſiquidem pro caſibus, quibus x eſt numerus integer poſitivus, tota res nulla difficultate laborat. Hoc igitur caſu ex forma ante data erit

$$\Sigma:x=\begin{cases}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.\\ -\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}-\frac{1}{x+4}-\&c.\end{cases}$$

quae duae ſeries in hanc unicam contrahentur

$$\Sigma:x=\frac{x}{x+1}+\frac{x}{2(x+2)}+\frac{x}{3(x+3)}+\frac{x}{4(x+4)}+\&c.$$

cuius ergo ſeriei ſumma per ſe conſtat, quoties x fuerit numerus

merus integer poſitivus, ita erit:

ſi $x = 1$	$1 = \frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \&c.$
$x = 2$	$1 + \frac{1}{2} = \frac{2}{1.3} + \frac{2}{2.4} + \frac{2}{3.5} + \frac{2}{4.6} + \&c.$
$x = 3$	$1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{1.4} + \frac{3}{2.5} + \frac{3}{3.6} + \frac{3}{4.7} + \&c.$
$x = 4$	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{4}{1.5} + \frac{4}{2.6} + \frac{4}{3.7} + \frac{4}{4.8} + \&c.$
&c.	&c.

quae quidem ſeries ſunt notiſſimae.

18. Quo haec clarius intelligantur conſtruamus curvam, cuius abſciſſae $ox = x$ reſpondeat applicata $xy = y = \Sigma : x$; ita ut ſumtis ſuper axe ox intervallis unitate aequalibus 0, 1; 1, 2; 2, 3; 3, 4; &c. applicatae futurae ſint

$1 \,..\, (1) = 1$

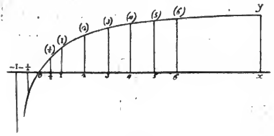
$2 \,..\, (2) = 1 + \frac{1}{2}$

$3 \,..\, (3) = 1 + \frac{1}{2} + \frac{1}{3}$

$4 \,..\, (4) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

&c. atque aequatio inter binas coordinatas erit

$$y = \frac{x}{x+1} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \&c.$$



ex qua ergo aequatione omnes applicatae intermediae definiri poterunt, atque adeo ſufficiet pro x valores unitate minores accepiſſe. Ita ſi applicata $\frac{1}{2}..(\frac{1}{2})$ abſciſſae $o..\frac{1}{2}=\frac{1}{2}$ deſideretur reperietur $\frac{1}{2}..(\frac{1}{2})=\frac{1}{3}+\frac{1}{2.5}+\frac{1}{3.7}+\frac{1}{4.9}+\frac{1}{5.11}+$ &c. cuius ſumma per logarithmos aſſignari poterit hoc modo: formetur haec ſeries: $y=\frac{t^3}{1.3}+\frac{t^5}{2.5}+\frac{t^7}{3.7}+\frac{t^9}{4.9}+$ &c. quae ergo ſeries ſumto $t=1$ dabit valorem quaeſitum, at vero differentiando habebimus:

$\frac{dy}{dt}=\frac{tt}{1}+\frac{t^4}{2}+\frac{t^6}{3}+\frac{t^8}{4}+$ &c. & denuo differentiando: $\frac{ddy}{2dt^2}=t+t^3+t^5+t^7+$ &c. $=\frac{t}{1-tt}$. Hinc ergo viciſſim erit $\frac{dy}{2dt}=\int\frac{t\,dt}{1-tt}$, & $y=2\int dt\int\frac{t\,dt}{1-tt}$, quae duplex integratio reducitur more ſolito ad unicam, quo facto erit $y=2t\int\frac{t\,dt}{1-tt}-2\int\frac{tt\,dt}{1-tt}$. Quia autem poſt integrationem ſtatui debet $t=1$, erit $y=2\int\frac{t\,dt}{1-tt}-2\int\frac{tt\,dt}{1-tt}=2\int\frac{t\,dt}{1+t}$; quamobrem integrando fiet $y=2t-2l(t+1)$, ideoque noſtro caſu $y=2-2l2$, cuius valor proxime verus eſt $0,61370564$.

19. Inventa iam applicata abſciſſae $\frac{1}{2}$ reſpondente, ſcilicet $\Sigma:\frac{1}{2}=2-2l2$, ex ea ſequentes per formulas ſupra datas facile derivantur; ſcilicet

$$\Sigma:(1+\tfrac{1}{2})=\frac{2}{3}+\Sigma:\tfrac{1}{2}$$

$$\Sigma:(2+\tfrac{1}{2})=\frac{2}{3}+\frac{2}{5}+\Sigma:\tfrac{1}{2}$$

$$\Sigma:(3+\tfrac{1}{2})=\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\Sigma:\tfrac{1}{2}$$

&c.

Quin etiam praecedentes applicatae in figura non expreſſae ex formula $\Sigma:(x-i)$ determinari poterunt, quam invenimus $\Sigma:(x-i)=\Sigma:x-(x)-(x-1)-(x-2)\ldots-(x-i+1)$ quia noſtro caſu $x=\frac{1}{2}$, erit applicata

$\Sigma:\left(-\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2=-2l2$, erit ſcilicet negativa, quae adeo ſumto $x=-1$ fit infinita. Infinita vero etiam evadet caſibus $x=-2$, $x=-3$, $x=-4$ &c.: intra autem haec intervalla erit

$$\Sigma:-\left(1+\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2+2$$

$$\Sigma:-\left(2+\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2+2-\frac{2}{3}$$

$$\Sigma:-\left(3+\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2+2-\frac{2}{3}+\frac{2}{5}$$

&c.

20. Differentiemus nunc ſeriem pro applicata y inventam fietque

$$\frac{dy}{dx}=\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\text{\&c.}$$

quae ergo ſeries exprimit tangentem anguli, ſub quo elementum curvae in y ad axem inclinatur; unde patet pro abſciſſa infinita hanc inclinationem fore nullam, ſive tractum curvae in infinito axi eſſe parallelum. Hinc ergo ſumto $x=0$ innoteſcet inclinatio curvae ad ipſum initium =

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{\&c.}=\frac{\pi\pi}{6}=1,644,$$

ideoque angulus $=58°, 42'$. Tum vero ſumto $x=1$ erit

$$\frac{dy}{dx}=\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{\&c.}=\frac{\pi\pi}{6}-1=0,644,$$

ubi ergo inclinatio erit $=32°, 48'$, hincque ulterius continuando inclinatio continuo decreſcet.

 21.

21. Retrogrediendo vero ad abſciſſas negativas ſupra vidimus, caſibus, quibus $x = -1$; vel $x = -2$; vel $x = -3$ applicatas fieri infinite magnas, & totidem curvae aſſymptotas conſtituere. Nunc vero videbimus, iiſdem locis fieri $\frac{dy}{dx} = \infty$, ibique inclinationem curvae eſſe 90°, ſive tangentes ad axem fore perpendiculares. Praeterea quoniam ſeries pro $\frac{dy}{dx}$ inventa ſemper habet ſummam poſitivam, ſequitur omnes partes curvae dextrorſum ſemper aſcendere; contra vero ſiniſtrorſum deſcendere.

22. Quin etiam poterimus integrationem adhibere, atque aream curvae ab initio uſque ad applicatam xy aſſignare. Ex prima enim forma, ad quam ſumus perducti immediate manifeſto fiet

$$\int y\,dx = \begin{matrix} x & + & \frac{1}{2}x & + & \frac{1}{3}x & + \&c. \\ -l(1+x) & - & l(2+x) & - & l(3+x) & - \&c. \end{matrix} + \text{Conſt.}$$

quae conſtans ita debet determinari, ut caſu $x = 0$ tota area evaneſcat, unde illa rite ita exprimetur

$$\int y\,dx = \begin{matrix} x & + & \frac{1}{2}x & + & \frac{1}{3}x & + \&c. \\ -l(1+x) & - & l(1+\frac{1}{2}x) & - & l(1+\frac{1}{3}x) & - \&c. \end{matrix}$$

Cum igitur ſit

$$l\left(1+\frac{x}{n}\right) = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} + \&c. - \&c.$$

ſuperior expreſſio per ſeries ſequentes exprimi poterit:

$$\begin{aligned} \int y\,dx = {} & \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^6}{6} - \&c. \\ & + \frac{x^2}{2.4} - \frac{x^3}{3.8} + \frac{x^4}{4.16} - \frac{x^5}{5.32} + \frac{x^6}{6.64} - \&c. \\ & + \frac{x^2}{2.9} - \frac{x^3}{3.27} + \frac{x^4}{4.81} - \frac{x^5}{5.243} + \frac{x^6}{6.729} - \&c. \\ & + \frac{x^2}{2.16} - \frac{x^3}{3.64} + \frac{x^4}{4.256} - \frac{x^5}{5.1224} + \frac{x^6}{6.4896} - \&c. \\ & \&c. \end{aligned}$$

23.

23. Quod ſi iam has ſeries verticaliter colligamus habebimus

$$\int y\,dx = \frac{1}{2}x^2\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\&c.\right)=+0{,}822467.x^2$$
$$-\frac{1}{3}x^3\left(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\&c.\right)=-0{,}400685.x^3$$
$$+\frac{1}{4}x^4\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\&c.\right)=+0{,}270581.x^4$$
$$-\frac{1}{5}x^5\left(1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1224}+\quad+\&c.\right)=-0{,}207385.x^5$$

Ponamus nunc $x=1$, ut prodeat area 01 (1), & quia fractiones decimales hic datae parum convergunt, notetur ſeriei cuiuſcunque, ubi ſigna alternantur, ſcilicet:

$$s = a - b + c - d + e - \&c.$$

ſummam per differentias continuas ita exprimi, ut ſit

$$s = \tfrac{1}{2}a - \tfrac{1}{4}\Delta a + \tfrac{1}{8}\Delta^2 a - \tfrac{1}{16}\Delta^3 a + \&c.$$

cuius ergo regulae ope calculus ſequenti modo inſtitui poterit:

	$-\Delta$	$+\Delta^2$	$-\Delta^3$
a) 0, 822467	0, 421782		
b) 0, 400685	0, 130104	0, 291678	
c) 0, 270581	0, 063196	0, 066908	0, 224770
d) 0, 207385	0, 037828	0, 025368	0, 041540
e) 0, 169557	0, 025507	0, 012321	0, 013047
f) 0, 144050	0, 018541	0, 006966	0, 005355
g) 0, 125509	0, 014175	0, 004366	0, 002600
h) 0, 111334	0, 011235	0, 002940	0, 001426
i) 0, 100099			

$+\Delta^4$	$-\Delta^5$	$+\Delta^6$	$-\Delta^7$	$+\Delta^8$	
0, 183230	0, 154737	0, 133936	0, 118072	0, 105564	&c.
0, 028493	0, 020801	0, 015864	0, 012508		
0, 007692	0, 004937	0, 003356			
0, 002755	0, 001581				
0, 001174					

24. Harum columnarum, quarum prima ex Calculi Differentialis Cap. VI. Part. II. pag. 365 eſt definita, numeri ſupremi referunt terminum primum a cum ſuis differentiis continuis, ſecundi vero deſcendendo referunt terminum b cum ſuis differentiis, tertii terminum c cum ſuis differentiis. Quia nunc ſupremi termini parum convergunt, duos primos $a - b$ actu colligamus, eritque $a - b = 0,421782$: ſequentium vero $c - d + e - f +$ &c. ſummam

$$= \frac{1}{2}c - \frac{1}{4}\Delta c + \frac{1}{8}\Delta^2 c - \frac{1}{16}\Delta^3 c + \&c.$$

ſecundum datam legem computemus eritque

$$\begin{aligned}
\tfrac{1}{2}c &= 0,135290 \\
-\tfrac{1}{4}\Delta c &= 0,015799 \\
+\tfrac{1}{8}\Delta^2 c &= 0,003171 \\
-\tfrac{1}{16}\Delta^3 c &= 0,000815 \\
+\tfrac{1}{32}\Delta^4 c &= 0,000220 \\
-\tfrac{1}{64}\Delta^5 c &= 0,000077 \\
+\tfrac{1}{128}\Delta^6 c &= 0,000026 \\
\text{ſeqq.} &= 0,000010 \\
\hline
\text{ſumma} &= 0,155408 \\
a - b &= 0,421782 \\
\hline
\text{Area} &= 0,577190
\end{aligned}$$

Spero autem, fuſiorem evolutionem huius Lineae Curvae ſatis memorabilis nemini fore ingratam praecipue cum aequatio pro hac curva pertineat ad functiones inexplicabiles, atque idcirco iſta ad caſum ſpecialiorem digreſſio a noſtro ſcopo haud aliena ſit exiſtimanda.

Species ſecunda ſerierum

quarum differentiae infiniteſimae primae evaneſcunt.

25. Ad hanc ergo ſpeciem pertinent omnes ſeries, quarum termini infiniteſimi inter ſe ſunt aequales. Ut ergo termi-

minum ſummatorium harum ſerierum $\Sigma : x$ exprimamus, nil aliud opus eſt, niſi ut ad expreſſionem praecedentis ſpeciei inſuper termini ſecundae columnae verticalis formae generalis §. 9. exhibitae adiungantur, cuius quidem terminus ſupremus ſeorſim erit exhibendus, & quia columnae ſingulae horizontales iam tribus terminis conſtant, terminus ſummatorius quaeſitus $\Sigma : x$ ſequenti ſerie triplicata definietur.

$$\Sigma : x = x(1) \left. \begin{matrix} + & (1) & - & (2) & + & (3) & - & (4) \\ + & x\Delta 1 & + & x\Delta 2 & + & x\Delta 3 & + & x\Delta 4 \\ - & (x+1) & - & (x+2) & - & (x+3) & - & (x+4) \end{matrix} \right\} \text{\&c.}$$

quae forma ob $\Delta 1 = (2) - (1)$; $\Delta 2 = (3) - (2)$; $\Delta 3 = (4) - (3)$; &c. transfundetur in hanc:

$$\Sigma : x = x(1) \left. \begin{matrix} + & \overline{1-x}(1) & + & \overline{1-x}(2) & + & \overline{1-x}(3) \\ + & x\ (2) & + & x\ (3) & + & x\ (4) \\ - & (x+1) & - & (x+2) & - & (x+3) \end{matrix} \right\} \text{\&c.}$$

quae ſeries eo magis convergit, quo minor x accipiatur. Supra autem docuimus omnes caſus ſemper eo reduci poſſe ubi x ſit fractio unitate minor.

26. Conſideremus primo caſum ſimpliciſſimum, quo omnes ſeriei termini ſunt inter ſe aequales, ſcilicet $(x) = a$; per ſe enim patet eius terminum ſummatorium eſſe ax, quem eundem valorem noſtra expreſſio ſtatim declarabit.

Erit enim $\Sigma\ x = xa$.

27. Nunc conſideretur caſus, quo $(x) = \frac{x+1}{x}$, ita ut noſtra ſeries ſit $\Sigma : x = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots + \frac{x+1}{x}$. cuius termini infiniteſimi omnes unitate aequantur. Noſtra igitur formula nobis dabit

$$\Sigma : x = 2x \left. \begin{matrix} + & \overline{1-x}\,\frac{2}{1} & + & \overline{1-x}\,\frac{3}{2} & + & \overline{1-x}\,\frac{4}{3} \\ + & x\,\frac{3}{2} & + & x\,\frac{4}{3} & + & x\,\frac{5}{4} \\ - & \frac{(x+2)}{x+1} & - & \frac{(x+3)}{x+2} & - & \frac{(x+4)}{x+3} \end{matrix} \right\} \text{\&c.}$$

unde patet ſumto $x = 1$ fore $\Sigma : x = \frac{2}{1}$; at ſumto $x = 2$ fiet:

$$\Sigma : x = 4 \left.\begin{matrix} -1.\frac{2}{1} - 1.\frac{3}{2} - 1.\frac{4}{3} \\ +2.\frac{3}{2} + 2.\frac{4}{3} + 2.\frac{5}{4} \\ -\frac{4}{3} - \frac{5}{4} - \frac{6}{5} \end{matrix}\right\} \&c. = 4 - \frac{2}{1} + \frac{3}{2}$$

28. At vero iſte caſus facile reduci poteſt ad ſpeciem praecedentem. Cum enim terminus generalis ſit $(x) = \frac{x+1}{x}$, is in partes reſolutus dabit $(x) = 1 + \frac{1}{x}$; quamobrem duae formentur ſeries, prior ſcilicet ex termino generali 1, altera vero ex termino generali $\frac{1}{x}$, haeque duae ſeries iunctim ſumtae dabunt ſummam quaeſitam $\Sigma : x$, erit ſcilicet

$$\Sigma : x = \begin{matrix} 1 + 1 + 1 + 1 + \ldots\ldots + x \\ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{x} \end{matrix}$$

Iam ſuperioris ſeriei ſumma eſt x, inferior vero per ſpeciem primam evolvi poteſt, indeque habebitur:

$$\Sigma : x = \begin{matrix} x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. \\ - \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} - \&c. \end{matrix}$$

quae expreſſio multo eſt ſimplicior praecedente, nihilo vero minus eundem valorem exhibet. Ita ſi ſumatur $x = \frac{1}{2}$ prior expreſſio nobis dabit $\Sigma : x = 1 \left.\begin{matrix} + \frac{1}{2}.\frac{2}{1} + \frac{1}{2}.\frac{3}{2} + \frac{1}{2}.\frac{4}{3} + \frac{1}{2}.\frac{5}{4} \\ + \frac{1}{2}.\frac{3}{2} + \frac{1}{2}.\frac{4}{3} + \frac{1}{2}.\frac{5}{4} + \frac{1}{2}.\frac{6}{5} \\ - \frac{5}{3} - \frac{7}{5} - \frac{9}{7} - \frac{11}{9} \end{matrix}\right\} \&c.$

ter-

terminiſque ſecundum ordinem collectis fiet

$$\Sigma : \tfrac{1}{2} = 1 + \frac{1}{3.4} + \frac{1}{5.12} + \frac{1}{7.24} + \frac{1}{9.40} + \frac{1}{11.60} + \&c.$$

cuius ordo clarius pateſcet ex ſequenti forma

$$\Sigma : \tfrac{1}{2} = 1 + \frac{1}{1.3.4} + \frac{1}{2.5.6} + \frac{1}{3.7.8} + \frac{1}{4.9.10} + \frac{1}{5.11.12} + \&c.$$

altera vero expreſſio dat has ſeries

$$\Sigma : \tfrac{1}{2} = \tfrac{1}{2} \begin{array}{l} + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c. \\ - \tfrac{2}{3} - \tfrac{2}{5} - \tfrac{2}{7} - \tfrac{2}{9} - \&c. \end{array}$$

quae collectae dabunt

$$\Sigma : \frac{1}{2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + \&c.$$

29. Ex hoc exemplo apparet, ſeriem ex ſpecie ſecunda deductam magis convergere quam poſteriorem ex ſpecie prima derivatam; quare operae pretium erit prioris ſeriei convergentiam attentius conſiderare. Quilibet ſcilicet huius ſeriei terminus oritur ex his tribus partibus

$$\frac{1}{2} \cdot \frac{n+1}{n} + \frac{1}{2} \cdot \frac{n+2}{n+1} - \frac{2n+3}{2n+1},$$

quae cum ſe mutuo proxime deſtruant, ſumma duorum priorum proxime aequalis erit tertiae, unde ſequitur haec formula ſatis memorabilis $\frac{n+1}{n} + \frac{n+2}{n+1} = \frac{2(n+3)}{2n+1}$, quod eo propius ad veritatem accedit, quo maior fuerit numerus n. Hinc utrinque ſubtrahendo 2 erit proxime

$$\frac{1}{n} + \frac{1}{n+1} = \frac{4}{2n+1}.$$

30. Talis autem reductio ad ſpeciem primam ſemper locum habere poteſt, quando ſeries propoſita tandem ad valorem finitum convergit; verum ſi ſeriei termini tandem in infinitum creſcant, haec reductio non amplius locum habere poteſt, ideoque neceſſario ad ſpeciem ſecundam erit recurrendum.

dum. Talis eſt caſus, quo $(x) = \sqrt{x}$, denotante enim n numerum infinitum bini termini infiniteſimi contigui erunt $\sqrt{n}$ & $\sqrt{n+1}$, quorum differentia eſt $\frac{1}{2\sqrt{n}}$, ideoque evaneſcens. Hoc ergo caſu ſeries noſtra erit

$$\Sigma : x = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \ldots\ldots\ldots + \sqrt{x}.$$

Hinc ergo per praecepta data habebimus hanc expreſſionem

$$\Sigma : x = x + \left.\begin{array}{l} +\,\overline{1-x}\sqrt{1} + \overline{1-x}\sqrt{2} + \overline{1-x}\sqrt{3} \\ \quad\; x\sqrt{2} \quad + \quad x\sqrt{3} \quad + \quad x\sqrt{4} \\ -\,\sqrt{x+1} - \sqrt{x+2} - \sqrt{x+3} \end{array}\right\} \&c.$$

quae ſeries quantopere convergat videamus caſu $x = \frac{1}{2}$, eritque

$$\Sigma : \tfrac{1}{2} = \tfrac{1}{2} + \left.\begin{array}{l} +\,\tfrac{1}{2}\sqrt{1} + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{4} \\ \;\;\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{4} + \tfrac{1}{2}\sqrt{5} \\ -\,\sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{5}{2}} - \sqrt{\tfrac{7}{2}} - \sqrt{\tfrac{9}{2}} \end{array}\right\} \&c.$$

cuius terminus quicunque erit $\frac{1}{2}\sqrt{n} + \frac{1}{2}\sqrt{n+1} - \sqrt{\frac{2n+1}{2}}$ qui eo propius ad nihilum accedere debet, quo maior fuerit numerus n, quocirca proxime erit $\sqrt{n} + \sqrt{n+1} = \sqrt{2(2n+1)}$. Sumtis enim quadratis habebimus $2n+1+2\sqrt{n(n+1)} = 2(2n+1)$, ideoque $2\sqrt{n(n+1)} = 2n+1$. Sumtis denuo quadratis fiet $4nn+4n = 4nn+4n+1$, quae ratio utique proxime ad aequalitatem accedit. Ceterum hic notari meretur, veros valores pro fractionibus loco x aſſumtis tantopere eſſe tranſcendentes, ut nullis plane formulis analyticis exprimi queant. Quin etiam quilibet valor pro x aſſumtus ad peculiare tranſcendentium genus pertinebit.

31. Antequam hanc ſpeciem deſeramus, adiungamus adhuc inſigne Theorema circa convergentiam formularum multo generalius eo, quod modo ante attulimus.

Theo-

Theorema

Sequens aequalitas

$$(\beta-\alpha)\sqrt[\mu]{n^\nu}+\alpha\sqrt[\mu]{(n+1)^\nu}=\beta\sqrt[\mu]{\left(n+\frac{\alpha}{\beta}\right)^\nu}$$

eo propius ad veritatem accedet, quo maior ſumatur numerus n, *ſimulque quo minor fuerit fractio* $\frac{\alpha}{\beta}$, *ſi modo exponens* $\frac{\nu}{\mu}$ *unitate fuerit minor. At vero ſumto* ν *negativo iſta aequalitas*

$$\frac{\beta-\alpha}{\sqrt[\mu]{n^\nu}}+\frac{\alpha}{\sqrt[\mu]{(n+1)^\nu}}=\frac{\beta}{\sqrt[\mu]{\left(n+\frac{\alpha}{\beta}\right)^\nu}}$$

ſine poſteriore conditione ad veritatem eo propius accedet, quo maior fuerit numerus n *&* *quo minor fuerit fractio* $\frac{\alpha}{\beta}$. *Quin etiam ſub iiſdem conditionibus ad logarithmos transferri poteſt, ita ut ſit tam*

$$(\beta-\alpha)\,l\,n+\alpha\,l(n+1)=\beta\,l\left(n+\frac{\alpha}{\beta}\right)$$

quam
$$\frac{\beta-\alpha}{l\,n}+\frac{\alpha}{l(n+1)}=\frac{\beta}{l\left(n+\frac{\alpha}{\beta}\right)}$$

Demonſtratio

32. Sequitur hoc Theorema ex ſolutione generali pro hac ſpecie data, cuius terminus quicunque conſiſtit his partibus $1-\varkappa(n)+\varkappa(n+1)-(n+\varkappa)$, atque eo minor evadit, quo maior ſumatur numerus n exiſtente $\varkappa$ fractione unitate minore. Quod ſi iam ponamus $\varkappa=\frac{\alpha}{\beta}$ & $(n)=\sqrt[\mu]{n^\nu}$, ideoque

etiam $(n) = \sqrt[\mu]{n^\nu}$, neceſſe eſt ut ſit $\frac{\mu}{\nu} < 1$, quia alioquin termini infiniteſimi non haberent differentias evaneſcentes. Hae autem ſubſtitutiones praebent formulas priores in theoremate datas. Quando vero fractio $\frac{\mu}{\nu}$ negativa accipiatur, tum ſeries propoſita adeo in ſpecie prima continebitur, ſiquidem ipſi termini infiniteſimi in nihilum abeunt.

33. Quo vis huius Theorematis clarius intelligatur, notaſſe iuvabit, has formulas quatuor caſibus exacte cum veritate convenire, quorum primus eſt ſi $\alpha = 0$; ſecundus quo $\alpha = 6$; tertius eſt quo $\nu = 0$; quartus denique locum habet ſi pro n accipiatur numerus infinitus; praeterea vero datur caſus quintus, quo in forma priore eſt $\mu = \nu$ ſive $\sqrt[\mu]{n^\nu} = n$.

Species tertia ſerierum

quarum differentiae demum infiniteſimae ſecundae evaneſcunt.

34. Hoc igitur eveniet, quoties ipſi termini infiniteſimi progreſſionem arithmeticam conſtituunt; formula igitur ante pro $\Sigma : x$ in ſuperiore ſpecie inventa ad hunc caſum accommodabitur, ſi inſuper ſinguli termini columnae tertiae verticalis adiungantur; hoc modo terminus ſummatorius ſequenti modo exprimetur

$$\Sigma : x = \begin{cases} x(1) + (1) + (2) + (3) \ldots\ldots + (n) \\ \quad + x\Delta 1 + x\Delta 2 + x\Delta 3 \ldots\ldots + x\Delta n \\ x'\Delta 1 + x'\Delta^2 1 + x'\Delta^2 2 + x'\Delta^2 3 \ldots\ldots + x'\Delta^2 n \\ \quad - (x+1) - (x+2) - (x+3) \ldots\ldots - (x+n) \end{cases}$$

35. Transmutemus nunc hanc expreſſionem in formam ad uſum magis accommodatam, ac primo quidem loco x' ſcribamus eius valorem $\frac{xx - x}{2}$; tum vero ob $\Delta n = (n+1) - (n)$

& $\Delta^2 n = (n+2) - 2(n+1) + (n)$, his valoribus ſubſtitutis poſtrema columna praecedentis formulae abibit in hanc formam:

$$(n) + x(n+1) + \frac{xx-x}{2}(n+2)$$
$$- x(n) - \overline{xx-x}\,(n+1)$$
$$+ \frac{xx-x}{2}(n)$$

qui termini collecti praebent

$$\frac{xx-3x+2}{2}(n) - \overline{xx-2x}\,(n+1) + \frac{xx-x}{2}(n+2).$$

Ponamus igitur brev. gr. $\frac{xx-3x+2}{2} = p$; $xx - 2x = q$ & $\frac{xx-x}{2} = r$, ſicque terminus ſummatorius quaeſitus ſequenti forma exprimetur:

$$\Sigma : x = \frac{3x-xx}{2}(1) + \frac{xx-x}{2}(2)$$
$$+ p(1) - q(2) + r(3) - (x+1)$$
$$+ p(2) - q(3) + r(4) - (x+2)$$
$$+ p(3) - q(4) + r(5) - (x+3)$$

quae ſeries iam vehementer converget.

36. Hinc igitur novum Theorema ſimile praecedenti, ſed multo latius patens poſſumus derivare, ponendo ut ante $x = \frac{\alpha}{6}$, $(n) = \sqrt[\mu]{n^\nu}$, ubi iam ſufficit ut exponens $\frac{\nu}{\mu}$ binario ſit minor, multo magis autem hunc exponentem negativum ſtatuere licebit. Theorema hoc eſt

Iſta aequalitas

$$(\alpha\alpha - 3\alpha 6 + 266)\sqrt[\mu]{n^\nu} - (2\alpha\alpha - 4\alpha 6)\sqrt[\mu]{(n+1)^\nu} + (\alpha\alpha - \alpha 6)\sqrt[\mu]{(n+2)^\nu} =$$
$$266\sqrt[\mu]{\left(n+\frac{\alpha}{6}\right)^\nu}$$

eo

eo propius ad veritatem accedet quo maior capiatur numerus n *&* *fractio* $\frac{\alpha}{\beta}$ *parum ab unitate diſcrepet, dummodo* $\frac{\nu}{\mu}$ *binario ſit minus. Tum vero ſumto* μ *negativo erit plerumque multo accuratius:*

$$\frac{\alpha\alpha-3\alpha\beta+2\beta\beta}{\sqrt[\mu]{n^\nu}}-\frac{2\alpha\alpha-4\alpha\beta}{\sqrt[\mu]{(n+1)^\nu}}+\frac{\alpha\alpha-\alpha\beta}{\sqrt[\mu]{(n+2)^\nu}}=\frac{2\beta\beta}{\sqrt[\mu]{(n+\frac{\alpha}{\beta})^\nu}}$$

Quin etiam pro formulis radicalibus logarithmi accipi poterunt.

37. Veritas huius Theorematis etiam exacte ſubſiſtit his quatuor caſibus I. $\alpha=0$; II. $\alpha=\beta$; III. $\nu=0$ & IV. $n=\infty$. Praeterea vero idem evenit, quando in forma priori eſt vel $\nu=\mu$, vel $\nu=2\mu$, ita ut ſit $\sqrt[\mu]{n^\nu}$ vel n vel nn. Habemus igitur ſex caſus, quibus hoc Theorema nihil plane a veritate aberrat, unde facile intelligitur etiam reliquis caſibus omnibus errorem non eſſe poſſe notabilem.

38. Poſſumus etiam hoc theorema adhuc generalius reddere, loco n ſcribendo $\frac{n}{c}$ & ubique per debitam poteſtatem ipſius c multiplicando, quo fractiones tollantur. Sicque prior forma fiet $(\alpha\alpha-3\alpha\beta+2\beta\beta)\sqrt[\mu]{n^\nu}-(2\alpha\alpha-4\alpha\beta)\sqrt[\mu]{(n+c)^\nu}$
$+(\alpha\alpha-\alpha\beta)\sqrt[\mu]{(n+2c)^\nu}=2\beta\beta\sqrt[\mu]{\left(n+\frac{\alpha c}{\beta}\right)^\nu}$
altera autem forma ab hac non diſcrepat, niſi quod radicalia in denominatorem ingrediuntur, id quod etiam de logarithmis eſt intelligendum.

39. Operae pretium erit hoc theorema aliquo exemplo illuſtrare. Sumatur igitur $\alpha=1$ & $\beta=2$, fientque aequalitates in theoremate exhibitae

$3\sqrt{n}$

$$3\sqrt[\mu]{n^\nu}+6\sqrt[\mu]{(n+c)^\nu}-\sqrt[\mu]{(n+2c)^\nu}=8\sqrt[\mu]{\left(n+\frac{1}{2}c\right)^\nu}$$

$$\frac{3}{\sqrt[\mu]{n^\nu}}+\frac{6}{\sqrt[\mu]{(n+1)^\nu}}-\frac{1}{\sqrt[\mu]{(n+2)^\nu}}=\frac{8}{\sqrt[\mu]{(n+\frac{1}{2})^\nu}}$$

Applicemus formam priorem ad logarithmos, fietque $3\,l\,n+6\,l\,(n+c)-l(n+2c)=8\,l\,(n+\frac{1}{2}c)$ ſit nunc $n=10$ & $c=2$, ut prodeat

$3\,l\,10+6\,l\,12-l\,14=8\,l\,11$. Facta igitur evolutione

$$3\,l\,10=3,\ 0000000 \qquad l\,14=1,\ 1461280$$
$$6\,l\,12=\underline{6,\ 4750872} \qquad 8\,l\,11=\underline{8,\ 3311416}$$
$$9,\ 4750872 \quad = \quad 9,\ 4772696$$

quarum differentia eſt 0, 0021824, quae multo minor prodiret, ſi numero n maiorem valorem tribuerimus

40. Circa ipſum autem terminum ſummatorium ſeriei propoſitae imprimis notari convenit, tam differentiationem quam integrationem facile inſtitui poſſe, ſumto ſcilicet indice x variabili, quemadmodum hoc iam in ſpecie prima fuſius eſt oſtenſum, ubi ipſe terminus ſummatorius $\Sigma : x$ tanquam applicata cuiuſdam curvae eſt conſideratus, dum index x referebat abſciſſam, hocque reſpectu in calculo differentiali potiſſimum functiones inexplicabiles ſum perſcrutatus.

41. Ex formula autem generali pro termino ſummatorio $\Sigma : x$ ſupra data evolvamus hic quoque caſum ſeriei harmonicae, quo eſt

$$\Sigma : x = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{x}$$

& quaeramus eius valorem pro indice $x=\frac{1}{2}$, atque ob

$$(x)=\frac{1}{x}\ ;\ p=\frac{3}{8}\ ;\ q=-\frac{3}{4}\ ;\ r=-\frac{1}{8}$$

habebimus

Σ:

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$$\Sigma : \tfrac{1}{2} = \frac{5}{8} - \frac{1}{16} \begin{array}{l} + \frac{3}{8} + \frac{3}{16} + \frac{1}{8} + \frac{3}{32} \\ + \frac{3}{8} + \frac{1}{4} + \frac{3}{16} + \frac{3}{20} \\ - \frac{1}{24} - \frac{1}{32} - \frac{1}{40} - \frac{1}{48} \\ - \frac{2}{3} - \frac{2}{5} - \frac{2}{7} - \frac{2}{9} \end{array} \Bigg\} \text{ \&c.}$$

Sive erit $8\,\Sigma : \frac{1}{2} = \frac{9}{2} \begin{array}{l} + \frac{3}{1} + \frac{3}{2} + \frac{3}{3} + \frac{3}{4} \\ + \frac{6}{2} + \frac{6}{3} + \frac{6}{4} + \frac{6}{5} \\ - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} \\ - \frac{16}{3} - \frac{16}{5} - \frac{16}{7} - \frac{16}{9} \end{array} \Bigg\} \text{ \&c.}$

Contrahamus ſingulas columnas in unam ſummam eritque

$$8\,\Sigma : \tfrac{1}{2} = \frac{6}{1.2.3.5} + \frac{6}{2.3.4.5} + \frac{6}{3.4.5.7} + \frac{6}{4.5.6.9} + \text{\&c.}$$

quae ſeries utique magis convergit ea, quam ſpecie ſecunda invenimus.

42. Quodſi autem terminos non contrahamus, ſed eos qui eundem habent denominatorem colligamus, omiſſa ſerie infima habebimus

$$8\,\Sigma : \tfrac{1}{2} = \frac{9}{2} + \frac{3}{1} + \frac{9}{2} \begin{array}{l} + 8\,(\tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \text{\&c.}) \\ - 16\,(\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \text{\&c.}) \end{array}$$

ſive loco ſuperioris ſeriei ſcribendo $16\,(\frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \text{\&c.})$ habebimus $\frac{1}{2}\,\Sigma : \frac{1}{2} - \frac{3}{4} = -\frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \frac{1}{11} + \text{\&c.}$ addamus utrinque $l\,2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{\&c.}$ fiet $\frac{1}{2}\,\Sigma : \frac{1}{2} - \frac{3}{4} + l\,2 = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$; conſequenter $\Sigma : \frac{1}{2} = 2 - 2\,l\,2$, qui valor egregie convenit cum eo, qui in ſpecie prima eſt datus.

SUP-

SUPPLEMENTUM
DE FUNCTIONIBUS INEXPLICABILIBUS FORMAE

$$\pi : x = A . B . C . D . E . \dots . X.$$

1. Hic factores A, B, C, D, &c. ſunt termini cuiuſdam ſeriei indicibus 1, 2, 3, 4, &c. reſpondentes, & X terminus indici x reſpondens; factores autem, qui indicibus ſequentibus $x+1$; $x+2$; $x+3$; &c. reſpondent per X', X'', X''' &c. deſignabo. Hinc iam ſtatim patet fore $\pi:(x+1) = X'. \pi:x$, & $\pi:(x+2) = X'. X''. \pi:x$, & ita porro. Praecedentes vero erunt $\pi:(x-1) = \frac{\pi:x}{X'}$, &c. Unde intelligitur ſufficere, dummodo hae formulae pro valoribus ipſius x unitate minoribus aſſignentur.

2. Quoties fuerit x numerus integer poſitivus, valores ipſius $\pi:x$ ſponte ſe produnt. Erit nempe

$\pi:1 = A$; $\pi:2 = AB$; $\pi:3 = ABC$; &c.

Quando autem x non eſt numerus integer poſitivus, productum, quod charactere $\pi:x$ deſignamus erit functio inexplicabilis ipſius x, niſi forte factores A, B, C, D, &c. ita fuerint comparati, ut praecedentes a ſequentibus deſtruantur, veluti evenit in hac forma: $\pi:x = \frac{1}{2}.\frac{2}{3}.\frac{3}{4}.\frac{4}{5} \dots \frac{x}{x+1}$: quandoquidem hic manifeſto eſt $\pi:x = \frac{1}{x+1}$, vel etiam in hoc exemplo: $\pi:x = \frac{3}{4}.\frac{8}{9}.\frac{15}{16}.\frac{24}{25} \dots \frac{xx+2x}{(x+1)^2}$; hinc enim erit

$$\pi:1 = \frac{3}{2.2};\ \pi:2 = \frac{2}{3} = \frac{4}{2.3};\ \pi:3 = \frac{5}{8} = \frac{5}{2.4};\ \pi:4 = \frac{3}{5} = \frac{6}{2.5};$$

$\pi : 5 = \frac{4}{2.6}$; &c. unde patet in genere fore $\pi : x = \frac{x+2}{2(x+1)}$.

3. Caſus autem inexplicabiles ſumendis logarithmis ad praecedentem diſſertationem revocabuntur. Erit enim

$$l\pi : x = lA + lB + lC \; . \; . \; . \; . \; + lX,$$

quae forma cum ſupra tractata comparata nobis dabit ſequentes valores

$\Sigma : x = l\pi : x$; $(1) = lA$; $(2) = lB$; $(3) = lC$; &c. & $(x) = lX$ tum vero erit $(x+1) = lX'$; $(x+2) = lX''$; &c. hocque conſenſu obſervato ſpecies ſupra tractatas ad praeſentem caſum accommodemus.

Species prima

ubi logarithmi factorum infiniteſimorum evaneſcunt, ſive ubi factores infiniteſimi unitati aequantur.

4. Cum igitur pro hac ſpecie introductis valoribus modo datis habeamus:

$$l\pi : x = \begin{matrix} lA + lB + lC + lD + \&c. \\ -lX' - lX'' - lX''' - lX^{IV} - \&c. \end{matrix}$$

ad numeros aſcendendo erit

$$\pi : x = \frac{A}{X'} \cdot \frac{B}{X''} \cdot \frac{C}{X'''} \cdot \frac{D}{X^{IV}} \cdot \&c.$$

Hic nulla exempla ſubiungo, quia iam plura in Calculo Differentiali ſunt evoluta.

Species ſecunda

ubi factores infiniteſimi inter ſe ſunt aequales.

5. Tum enim eorum logarithmi etiam inter ſe erunt aequales, ideoque differentiae primae omnes evaneſcent. Huc igitur

igitur accommodemus formam ſupra §. 25. inventam eritque

$$l\pi:x = x\,lA \begin{array}{l} +\overline{1-x}\,lA+\overline{1-x}\,lB+\overline{1-x}\,lC \\ +\quad x\,lB+\quad x\,lC+\quad x\,lD \\ -\quad lX'-\quad lX''-\quad lX''' \end{array}\Bigg\}\ \&c.$$

unde ad numeros aſcendendo habebimus.

$$\pi:x = A^{x}.\frac{A^{1-x}B^{x}}{X'}.\frac{B^{1-x}C^{x}}{X''}.\frac{C^{1-x}D^{x}}{X'''}.\ \&c.$$

Species tertia

ubi termini infiniteſimi conſtituunt progreſſionem geometricam.

6. Tum enim logarithmi horum terminorum progreſſionem arithmeticam conſtituent, cuius ergo differentiae ſecundae evaneſcent. Ut iam expreſſionem ſupra §. 35. inventam ad hunc caſum accommodemus, notandum eſt br. gr. poſitum fuiſſe $p = \frac{xx-3x+2}{2}$; $q = xx-2x$; & $r = \frac{xx-x}{2}$, unde habebimus

$$l\pi:x = \left.\begin{array}{r} +p\,lA+p\,lB+p\,lC \\ \frac{3x-xx}{2}\,lA-q\,lB-q\,lC-q\,lD \\ \frac{xx-x}{2}\,lB+r\,lC+r\,lD+r\,lE \\ -lX'-lX''-lX''' \end{array}\right\}\ \&c.$$

Ponamus autem hic porro compendii cauſſa $\frac{xx-3x}{2} = m$ & $\frac{xx-x}{2} = n$, atque ad numeros aſcendendo habebimus hanc expreſſionem.

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$$\pi : x = \frac{B^n}{A^m} \cdot \frac{A^p C^r}{B^q X'} \cdot \frac{B^p D^r}{C^q X''} \cdot \frac{C^p E^r}{D^q X'''} \cdot \&c.$$

7. Hoc modo confido, doctrinam de Functionibus Inexplicabilibus, quae in Calculo Differentiali non ſatis accurate & luculenter eſt expoſita, fere penitus exhauſiſſe, ita ut nihil amplius deſiderari poſſit, id quod eo magis neceſſarium videbatur, cum hoc argumentum plane ſit novum & a nemine adhuc tractatum, praecipue autem eius ſummus uſus in interpolatione ſerierum, atque hinc adeo ſymptomata linearum curvarum, quarum applicatae per functiones inexplicabiles exprimuntur inveſtiganda erant.

AD-

ADNOTATIONES

ADNOTAT. poſt Cap. II. Part. I.

1.

Propoſita ſit differentialis aequatio $dy + yXdx = Zdx$, in qua X & Z functiones quaſcunque exprimunt variabilis x: conſtat huius aequationis integrationem obtineri ponendo $y = uz$, ex quo oritur $udz + zdu + uzXdx = Zdx$, ubi per idoneum quantitatis u, vel z valorem poſſunt bini termini nihilo aequari. Aſſumamus igitur $zdu + uzXdx = 0$; & dividendo per z fiet $du + uXdx = 0$, & conſequenter $\frac{du}{u} = -Xdx$, atque hinc capto integrali prodibit $lu = -\int Xdx$, hoc eſt $u = e^{-\int Xdx}$, ſumto nimirum e aequali logarithmorum hyperbolicorum baſi. Hiſce peractis propoſita aequatio convertitur in $udz = Zdx$, unde habetur $dz = \frac{Zdx}{u}$, & integrando $z = \int \frac{Zdx}{u} = \int e^{\int Xdx} Zdx$, ac denique $y = uz = \frac{\int e^{\int Xdx} Zdx}{e^{\int Xdx}}$

2. Si nunc methodus haec diligenter perpendatur, liquido apparebit, eandem felici exitu ad aequationes illas differentiales transferri poſſe, quae eandem habentes formam, quam aequatio praecedens, finitis tamen differentiis donantur. Eſto igitur aequatio $\Delta y + My\Delta x = N\Delta x$, ſeu $\Delta y + My = N$ (ſumto nimirum Δx pro unitate), in qua quantitates M & N functiones deſignant variabilis cuiuslibet x. Fiat primo $y = uz$, & erit in hac differentiarum finitarum hypotheſi

$$\Delta y = u\Delta z + z\Delta u + \Delta u\Delta z;$$

atque hinc aequatio abibit in

$$u\Delta z + z\Delta u + \Delta u\Delta z + Muz = N.$$

Po-

Ponamus ut ante terminos binos $z\Delta u + Muz = 0$, & orietur $\Delta u + Mu = 0$, ſeu $\frac{\Delta u}{u} = -M$: ad hanc aequationem in hac hypotheſi differentiae finitae Δu integrandam aſſumo $u = e^t$, & nanciſcor $u + \Delta u = e^{t + \Delta t}$, & $\Delta u = e^t\,(e^{\Delta t} - 1)$; ex quo fit $\frac{\Delta u}{u} = e^{\Delta t} - 1 = -M$, ſeu $e^{\Delta t} = 1 - M$, captiſque logarithmis $\Delta t = l(1 - M)$, & conſequenter inſtituta integratione habebimus $t = \Sigma\, l(1 - M)$. Eſt autem, ut ex Analyſi conſtat, aggregatum ex logarithmis plurium numerorum aequale logarithmo producti horum omnium numerorum; ſi igitur generatim per $\pi(1 - M)$ exprimamus productum continuum quantitatum omnium contentarum in formula $1 - M$, orietur $t = l\,\pi(1 - M)$; & propterea $u = e^t = \pi(1 - M)$. Iam praedictis duobus terminis nulleſcentibus ſuperior aequatio mutatur in $u\Delta z + \Delta u \Delta z = N$, unde eruitur $\Delta z = \frac{N}{u + \Delta u}$, & integrando $z = \Sigma\, \frac{N}{u + \Delta u}$. Sed cum iam invenerimus $u = \pi(1 - M)$, ſi terminus poſt M proxime ſubſequens deſignetur per M', prodit $u + \Delta u = \pi(1 - M')$; ideoque $z = \Sigma\, \frac{N}{\pi(1 - M')}$, & quia $y = zu$, fit

$$y = \pi(1 - M)\,\Sigma\, \frac{N}{\pi(1 - M')},$$

ſive addita conſtante quacunque A conſequemur

$$y = \pi(1 - M)\left(A + \Sigma\, \frac{N}{\pi(1 - M')}\right).$$

EXEMPLUM.

Sit propoſita aequatio $y + (x + 1)\Delta y + a(2x + 1) = 0$, *quaeraturque valor ſeparatus ipſius* y.

Haec aequatio ad generalem formam $\Delta y + My = N$ reducta dat $\Delta y + \frac{1}{x + 1}\,y = -\frac{a(2x + 1)}{x + 1}$. Habemus ergo M =

$M = \frac{1}{x+1}$; $N = -\frac{a(2x+1)}{x+1}$; $1 - M = \frac{x}{x+1}$; quapropter productum omnium valorum formulae $\frac{x}{x+1}$, qui habentur ſi in ea loco x ſucceſſive $x-1, x-2, \ldots 3, 2, 1$, ſubſtituantur, ſcilicet $\frac{x-1}{x} \cdot \frac{x-2}{x-1} \cdot \frac{x-3}{x-2} \cdots \frac{1}{2}$, & per $\pi(1-M)$ indicavimus, erit $= \frac{1}{x}$; illud vero per $\pi(1-M')$ expreſſum $= \frac{1}{x+1}$. Hinc hiſce valoribus in aequatione $y = \pi(1-M)\left(A + \Sigma \frac{N}{\pi(1\text{-}M')}\right)$ ſubſtitutis habebitur $y = \frac{1}{x}\left[A - \Sigma\, a(2x+1)\right] = \frac{A}{x} - 2a\frac{\Sigma x}{x} - a\frac{\Sigma 1}{x}$.

Inventum eſt autem §. 60. $\Sigma x^0 = \Sigma 1 = x$, & $\Sigma x = \frac{1}{2}x^2 - \frac{1}{2}x$ proindeque erit $y = \frac{A}{x} - ax$

3. Sit modo propoſita aequatio $y' = Ry + T$, in qua y' terminum deſignat, qui in ſerie quantitatum y proxime ſequitur y: hinc quoniam $y' = y + \Delta y$, aequatio abit in $\Delta y + (1-R)y = T$. Inſtituta huius aequationis cum praecedenti comparatione prodit $1 - R = M$; $T = N$. Quapropter pro valore quantitatis y ſequens habetur expreſſio

$$y = \pi R\left(A + \Sigma \frac{T}{\pi R'}\right).$$

Si R quantitas conſtans eſt, perſpicuum fit quantitates πR & $\pi R'$ nihil aliud eſſe quam potentias ipſius R, quarum exponens aequatur numero deſignanti in ſerie ipſarum y locum vel indicem terminorum y & y'. Sit igitur m numerus iſte, vel index loci ab y occupati ita ut y^m idem ſit ac y', & habe-

habebitur aequatio $y^m = R^m \left(A + \Sigma \frac{T}{R^{m+1}}\right)$. Si T conſtans eſt, fit $\Sigma \frac{T}{R^{m+1}} = T\,\Sigma \frac{1}{R^{m+1}}$, ubi termini per $\frac{1}{R^{m+1}}$ exhibiti progreſſionem geometricam conſtituunt, cuius ſumma protinus invenitur: ſumma haec ab $\frac{1}{R}$ incipiens, vocetur S, ponatur nimirum

$$\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \ldots\ldots + \frac{1}{R^m} = S,$$

& multiplicando per R conſequemur

$$1 + \frac{1}{R} + \frac{1}{R^2} + \ldots\ldots + \frac{1}{R^{m-1}} = SR = S + 1 - \frac{1}{R^m}.$$

Ex hac aequatione elicitur $S = \frac{R^m - 1}{R^m(R-1)}$; atque inde

$$y^m = R^m \left(A + T.\frac{R^m - 1}{R^m(R-1)}\right), \text{ ſive } y^m = AR^m + T.\frac{R^m - 1}{R - 1}.$$

Ut nunc oſtendamus per inventum ipſius y valorem omnimode ſatisfieri conditionibus datae aequationis $y' = Ry + T$, vel $y^{m+1} = Ry^m + T$, non alia re opus erit quam multiplicatione inventae formulae per R, & additione quantitatis T, ex quo colligitur expreſſio $AR^{m+1} + T.\frac{R^{m+1} - R}{R - 1} + T$, quae redigitur ad $AR^{m+1} + T.\frac{R^{m+1} - 1}{R - 1}$, hicque revera eſt valor, quem generalis formula pro termino y^{m+1} ſuppeditat.

4. Tradita iam methodo integrandi aequationem quamcumque differentialem ex differentiis finitis conſtantem, & ſub generali forma $\Delta y + My = N$ comprehenſam, reliquum eſt ut ad integrandas aequationes alias ab hac eadem pendentes progrediamur. Iam vero demonſtravit ALEMBERTUS in Actis Beroli-

Berolinenſis Academiae, aequationes omnes differentiales e differentiis infiniteſimis conſtantes huiuſce formae

$$(A) \qquad y + \frac{A dy}{dx} + \frac{B ddy}{dx^2} + \frac{C d^3 y}{dx^3} + \&c. = X.$$

ubi A, B, C, &c. conſtantes ſunt, & X functio eſt quaecunque ipſius x, revocari & converti in aequationem hanc ſimpliciorem $z + \frac{H dz}{dx} = V$, ubi H conſtans eſt, & V functio ipſius x; quae ſane aequatio eadem eſt cum ea, quam integrare docuimus ſuppoſitis etiam differentiis finitis. Si igitur Alembertiana methodus aequationibus etiam differentiarum finitarum accommodari poteſt licebit quoque integrare in hac ipſa differentiarum finitarum hypotheſi aequationem quamlibet

$$y + A\Delta y + B\Delta\Delta y + C\Delta^3 y + \&c. = X,$$

& conſequenter aequationem huius formae

$$y^{\text{I}} + P y^{\text{II}} + Q y^{\text{III}} \ldots + \&c. = X$$

quae pro generali ſerierum recurrentium formula haberi merito poteſt. Ingenioſiſſima Alembertiana methodus, quae *coefficientium indeterminatorum* methodus appellatur, in hoc eſt poſita: Sumitur ſcilicet

$$\frac{dy}{dx} = p \;;\; \frac{dp}{dx} = q \;;\; \frac{dq}{dx} = r \;;\; \&c.$$

indeque fiunt aequationes

$$p - \frac{dy}{dx} = 0 \;;\; q - \frac{dp}{dx} = 0 \;;\; r - \frac{dq}{dx} = 0 \;;\; \&c.$$

Tum aequationes hae ſingulae in coefficientes indeterminatos a, b, c, &c. ducuntur, ut oriatur

$$ap - \frac{ady}{dx} = 0 \;;\; bq - \frac{bdp}{dx} = 0 \;;\; cr - \frac{cdq}{dx} = 0 \;;\; \&c.$$

Hae vero poſtremae addantur aequationi (A), quae ſi ultra tertias differentias variabilis y non procedat, mutatur factis debitis ſubſtitutionibus in ſequentem

$$(B) \ldots y+(A+a)p+(B+b)q-\frac{ady}{dx}-\frac{bdp}{dx}+\frac{Cdq}{dx}=X.$$

In primo huius aequationis membro ſumamus nunc partem unam $y+(A+a)p+(B+b)q$ ductam in a eſſe multiplam integralis ex parte altera $ady+bdp-Cdq$ reſultantis, vel quod eodem redit eſſe

$$dy+(A+a)dp+(B+b)dq=dy+\frac{bdp}{a}-\frac{Cdq}{a};$$

unde ex terminorum ſibi reſpondentium comparatione prodeunt aequalitates $A+a=\frac{b}{a}$; $B+b=-\frac{C}{a}$, ex quibus conſequimur $b=-\frac{C}{a}-B=Aa+a^2$, & $a^3+Aa^2+Ba+C=0$. Huius porro aequationis $a^3+Aa^2+Ba+C=0$ radices praebent valores tres ipſius a diverſos, qui requiſitis conditionibus aeque ſatisfaciunt. Fiat nunc

$$y+(A+a)p+(B+b)q=z,$$

& aequatio inventa (B) mutabitur in

$$z-\frac{adz}{dx}=X, \text{ ſeu } dz-\frac{zdx}{a}=-\frac{Xdx}{a}.$$

Hac comparata cum §. 1. aequatione $dy+yPdx=Zdx$ (ubi ad ambiguitatem vitandam X convertimus in P) habetur

$$y=z;\ P=-\frac{1}{a};\ Z=-\frac{X}{a};\ e^{\int Pdx}=e^{\int-\frac{dx}{a}}=e^{-\frac{x}{a}},$$

ex quo demum colligitur $z=e^{\frac{x}{a}}\int\frac{-X\,dx}{ae^{\frac{x}{a}}}$. Voco nunc a^{I}, a^{II}, a^{III} tres diſtinctos ipſius a valores, & b^{I}, b^{II}, b^{III} valores alios ipſius b prioribus homologos, ac denique Z^{I}, Z^{II}, Z^{III} valores quantitatis variabilis z, quae per vices complectitur a^{I}, a^{II}, a^{III}. Hinc tres ſequentes prodibunt aequationes

$$\begin{aligned} y+(A+a^{I})p+(B+b^{I})q&=Z^{I}\\ y+(A+a^{II})p+(B+b^{II})q&=Z^{II}\\ (I)\qquad y+(A+a^{III})p+(B+b^{III})q&=Z^{III}\end{aligned}$$

Si

Si iam ex tribus hiſce aequationibus eliminantur quantitates p & q, invenitur ipſius y valor ſequenti aequatione exhibitus

$$y = FZ^{I} + GZ^{II} + HZ^{III}$$

ubi F, G, H conſtantes ſunt a quantitatibus A, B, a^{I}, a^{II}, &c. pendentes.

5. Hac methodo probe perpenſa luculenter apparet aequationem quamlibet multo pluribus terminis conflatam, cuiuſmodi eſſet iſthaec

$$y + \frac{A\,dy}{dx} + \frac{B\,ddy}{dx^2} + \frac{C\,d^3y}{dx^3} + \frac{D\,d^4y}{dx^4} + \frac{E\,d^5y}{dx^5} = X$$

aeque reſolvi poſſe, indeque obtineri

$$y = FZ^{I} + GZ^{II} + HZ^{III} + IZ^{IV} + KZ^{V},$$

ubi quantitates Z^{I}, Z^{II}, &c. tales ſunt ipſarum X & x functiones, ut ſit $Z = e^{\frac{x}{a}} \int \frac{-X\,dx}{a e^{\frac{x}{a}}}$, ſubſtitutis pro a radicibus quinque a^{I}, a^{II}, a^{III}, a^{IV}, a^{V} aequationis

$$a^5 + Aa^4 + Ba^3 + Ca^2 + Da + E = 0.$$

Sed quod longe utilius eſt ac praeſtantius methodus eadem ad aequationes quoque ex finitis differentiis conſtantes feliciſſime aptari poteſt.

6. Eſto itaque aequatio differentialis finita quinti ordinis

$$y + A\Delta y + B\Delta^2 y + C\Delta^3 y + D\Delta^4 y + E\Delta^5 y = X$$

& ponatur $\Delta y = p$; $\Delta p = q$; $\Delta q = r$; $\Delta r = s$, ut prodeant aequalitates

$$p - \Delta y = 0;\; q - \Delta p = 0;\; r - \Delta q = 0;\; s - \Delta r = 0$$

hiſque multiplicatis per coefficientes indeterminatos a, b, c, d fiat $ap - a\Delta y = 0$; $bq - b\Delta p = 0$; $cr - c\Delta q = 0$; $ds - d\Delta r = 0$. Aequationi propoſitae poſt ſubſtitutionem valorum aſſumtorum pro differentiis ipſius y addantur aequalitates modo inventae; quo facto eruitur aequatio

$$y + (A + a)p + (B + b)q + (C + c)r + (D + d)s$$
$$- a\Delta y - b\Delta p - c\Delta q - d\Delta r + E\Delta s = X$$

huius aequationis pars

 $y+$

$$y+(A+a)p+(B+b)q+(C+c)r+(D+d)s$$

differentiata aequalis aſſumatur parti alteri

$$a\Delta y+b\Delta p+c\Delta q+d\Delta r-E\Delta s$$

per $-a$ diviſae, ſcilicet fiat

$$\Delta y+(A+a)\Delta p+(B+b)\Delta q+(C+c)\Delta r+(D+d)\Delta s= \Delta y+\frac{b\Delta p}{a}+\frac{c\Delta q}{a}+\frac{d\Delta r}{a}-\frac{E\Delta s}{a}$$

unde per terminorum homologorum comparationem inferuntur aequalitates

$$A+a=\frac{b}{a};\ B+b=\frac{c}{a};\ C+c=\frac{d}{a};\ D+d=-\frac{E}{a},$$

ex quibus per conſuetas eliminationes exurgit aequatio quinti gradus

$$a^5+Aa^4+Ba^3+Ca^2+Da+E=0:$$

huius radices praebent quinque ipſius a diſtinctos valores a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}. Iam vero capiatur

$$y+(A+a)p+(B+b)q+(C+c)r+(D+d)s=z$$

ideoque

$$\Delta z=\Delta y+(A+a)\Delta p+(B+b)\Delta q+(C+c)\Delta r+(D+d)\Delta s= \Delta y+\frac{b}{a}\Delta p+\frac{c}{a}\Delta q+\frac{d}{a}\Delta r-\frac{E}{a}\Delta s,$$

unde oritur aequatio

$$z-a\Delta z=X,\ \text{hoc eſt}\ \Delta z-\frac{z}{a}=-\frac{X}{a}$$

eiuſdem ſcilicet formae cum praecedenti $\Delta y+My=N$ (§. 2.) ubi ex terminorum homologorum comparatione deducitur

$$M=-\frac{1}{a};\ N=-\frac{X}{a},\ \&\ \text{conſequenter}\ 1-M=\frac{1+a}{a}.$$

Quapropter oritur $z=\pi\left(\frac{1+a}{a}\right)\left(\text{Conſt.}+\Sigma\frac{-X}{a.\pi\left(\frac{1+a}{a}\right)}\right)$;

Et quoniam a conſtans eſt, invenitur

z

$$z^m = \left(\frac{1+a}{a}\right)^m \left(\text{Conſt.} - \Sigma \frac{X a^m}{(1+a)^{m+1}}\right)$$

denotante m ut antea indicem loci a termino z occupati in ſerie quantitatum z. Si X conſtans eſt tunc progreſſionis geometricae per $\Sigma \frac{a^m}{(1+a)^{m+1}}$ expreſſae accepta ſumma aſſequimur $z^m = \left(\frac{1+a}{a}\right)^m \left(\text{Conſt.} - X \frac{(1+a)^m - a^m}{(1+a)^m}\right)$.

Quum autem a quinque valoribus diverſis $a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}$ praeditus ſit, ſi valores iſti in formula inventa ſubrogantur per vices, totidem exſurgunt ipſius z^m valores, qui omnes aeque ſatisfacient; his vero per $Z^{I}, Z^{II}, Z^{III}, Z^{IV}, Z^{V}$ deſignatis quinque ſequentes nanciſcimur aequationes

$$y + (A + a^{I})p + (B + b^{I})q + (C + c^{I})r + (D + d^{I})s = Z^{I}$$
$$y + (A + a^{II})p + (B + b^{II})q + (C + c^{II})r + (D + d^{II})s = Z^{II}$$
$$y + (A + a^{III})p + (B + b^{III})q + (C + c^{III})r + (D + d^{III})s = Z^{III}$$
$$y + (A + a^{IV})p + (B + b^{IV})q + (C + c^{IV})r + (D + d^{IV})s = Z^{IV}$$
$$y + (A + a^{V})p + (B + b^{V})q + (C + c^{V})r + (D + d^{V})s = Z^{V}$$

Tum ex hiſce aequationibus per notas regulas eliminantur quantitates p, q, r, s, quo facto ad aequationem pervenitur huius formae $y = FZ^{I} + GZ^{II} + HZ^{III} + IZ^{IV} + KZ^{V}$, in qua F, G, H, I, K conſtantes ſunt a cognitis quantitatibus aequationum earundem pendentes.

7. Propoſita demum ſit aequatio

$$y^{I} + Ay^{II} + By^{III} + Cy^{IV} \ldots + \&c. = X$$

ubi y^{I}, y^{II}, y^{III}, &c. terminos deſignant ſe proxime ſequentes in ſerie quantitatum y. Perſpicuum imprimis eſt haberi

$$y^{II} = y^{I} + \Delta y$$
$$y^{III} = y^{II} + \Delta y^{II} = y^{I} + \Delta y^{I} + \Delta(y^{I} + \Delta y^{I}) = y^{I} + 2\Delta y^{I} + \Delta^2 y^{I}$$
$$y^{IV} = y^{III} + \Delta y^{III} = y^{I} + 2\Delta y^{I} + \Delta^2 y^{I} + \Delta(y^{I} + 2\Delta y^{I} + \Delta^2 y^{I})$$
$$= y^{I} + 3\Delta y^{I} + 3\Delta^2 y^{I} + \Delta^3 y^{I},$$

ſicque porro de ceteris. Ex quo apparet aequationem propoſitam ad formam earum, quas ante examinavimus, commo-

de

de revocari. Verum ut facilius ac promptius aequatio ipſa ad ſeries recurrentes accommodari queat praeſtabit terminos y^{I}, y^{II}, y^{III}, &c. ordine inverſo ſpectare, ut ſcilicet habeatur $y^{II} + \Delta y^{II} = y^{I}$; $y^{III} + \Delta y^{III} = y^{II}$; $y^{IV} + \Delta y^{IV} = y^{III}$, &c. & exponentes I, II, III, &c. deſignent cuiuslibet termini diſtantiam a poſtremo y^{I}. Ponatur $y^{II} = p^{I}$, eritque $y^{III} = p^{II}$: fiat iterum $p^{II} = q^{I}$, proindeque $p^{III} = q^{II}$; ſit rurſus $q^{II} = r^{I}$, ideoque $q^{III} = r^{II} = s^{I}$. Ex quo oriuntur aequalitates $y^{II} = p^{I}$; $y^{III} = q^{I}$; $y^{IV} = r^{I}$; $y^{V} = s^{I}$; $y^{VI} = s^{II}$: quibus valoribus in aequatione propoſita ſubrogatis ea abit in hanc (A) .... $y^{I} + Ap^{I} + Bq^{I} + Cr^{I} + Ds^{I} + Es^{II} = X$. Iam vero aequationibus $p^{I} - y^{II} = 0$; $q^{I} - p^{II} = 0$; $r^{I} - q^{II} = 0$; $s^{I} - r^{II} = 0$ per indeterminatos coefficientes a, b, c, d ſingillatim multiplicatis, hiſque de more additis aequationi (A) invenitur ſequens

$$\begin{matrix} y^{I} + (A+a)p^{I} + (B+b)\,q^{I} + (C+c)\,r^{I} + (D+d)s^{I} \\ - ay^{II} - bp^{II} - cq^{II} - dr^{II} + Es^{II} \end{matrix} = X$$

In prima huius aequationis parte

$$y^{I} + (A+a)p^{I} + (B+b)q^{I} + (C+c)r^{I} + (D+d)s^{I}$$

variantes termini tranſeant in proxime antecedentes ita ut fiat

$$y^{II} + (A+a)p^{II} + (B+b)q^{II} + (C+c)r^{II} + (D+d)s^{II}$$

haecque quantitas ponatur aequalis parti alteri per $-a$ diviſae

ſeu $$y^{II} + \frac{b}{a}p^{II} + \frac{c}{a}q^{II} + \frac{d}{a}r^{II} - \frac{E}{a}s^{II}$$

tum aequatis coefficientibus homologis habentur aequationes

$$A + a = \frac{b}{a};\ B + b = \frac{c}{a};\ C + c = \frac{d}{a};\ D + d = -\frac{E}{a},$$

ex quibus eruitur ut ante quinti gradus aequatio

$$a^{5} + Aa^{4} + Ba^{3} + Ca^{2} + Da + E = 0$$

cuius radices deſignantur per $a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}$. Quocirca accepto $z^{I} = y^{I} + (A+a)p^{I} + (B+b)q^{I} + (C+c)r^{I} + (D+d)s^{I}$, aequatio propoſita tranſibit in hanc $z^{I} - az^{II} = X$, quae (ob $z^{I} = z^{II} + \Delta z^{II}$) evadet $\Delta z^{II} + (1-a)z^{II} = X$. Hac autem comparata cum aequatione §. 3. oritur $y = z^{II}$,

R

$R = a$, $T = X$; ſitque propterea $z^m = a^m\left(\text{Conſt.} + \Sigma \frac{X}{a^{m+1}}\right)$ ubi m locum denotat a termino z^m in ſerie ipſarum z occupatum. Quum autem loco ipſius a ſubſtitui poſſint per vices radices ſingulae a^{I}, a^{II}, &c. inde conſequitur, quantitatis z^m valores quinque diverſos per Z^{I}, Z^{II} Z^{III}, Z^{IV}, Z^{V} expreſſos oriri, proindeque aequationes quinque ſequentes produci

$$y^m + (A + a^{I})p^m + (B + b^{I})q^m + (C + c^{I})r^m + (D + d^{I})s^m = Z^{I}$$
$$y^m + (A + a^{II})p^m + (B + b^{II})q^m + (C + c^{II})r^m + (D + d^{II})s^m = Z^{II}$$
$$y^m + (A + a^{III})p^m + (B + b^{III})q^m + (C + c^{III})r^m + (D + d^{III})s^m = Z^{III}$$
$$y^m + (A + a^{IV})p^m + (B + b^{IV})q^m + (C + c^{IV})r^m + (D + d^{IV})s^m = Z^{IV}$$
$$y^m + (A + a^{V})p^m + (B + b^{V})q^m + (C + c^{V})r^m + (D + d^{V})s^m = Z^{V}.$$

Eliminatis porro quantitatibus p^m, q^m, r^m, s^m ex hiſce aequationibus pervenitur ad formulam

$$y^m = FZ^{I} + GZ^{II} + HZ^{III} + IZ^{IV} + KZ^{V},$$

in qua F, G, H, &c. conſtantes ſunt, quae per comparationem totidem terminorum ſeriei ipſarum y determinari debent.

8. Si X conſtans eſt ſumma (§. 7.) per $\Sigma \frac{X}{a^{m+1}}$ expreſſa evadet $= X\frac{a^m - 1}{a^m(a-1)}$; dictaque L conſtanti integrationi addenda nanciſcimur tandem $Z = La^m + X\frac{a^m - 1}{a - 1}$, ex qua protinus habentur valores Z^{I}, Z^{II}, Z^{III}, &c. ſubſtitutis dumtaxat loco ipſius a radicibus inventis a^{I}, a^{II}, a^{III}, &c.

9. Ex hactenus dictis hoc ſtatim deducitur generale theorema:

Propoſita aequatione $y^m + Ay^{m-1} + By^{m-2} + Cy^{m-3} \ldots + \&c. = X$, *ubi ipſius* y *exponentes deſignant loca ab* y *occupata; inventiſque radicibus* a^{I}, a^{II}, a^{III} *&c. aequationis*

$$a^n + Aa^{n-1} + Ba^{n-2} + \&c. = 0,$$

habebitur generatim

y^m

$$y^m = Fa^{Im}\left(\psi + \Sigma \frac{X}{a^{Im+1}}\right) + Ga^{IIm}\left(\psi + \Sigma \frac{X}{a^{IIm+1}}\right)$$

$$+ Ha^{IIIm}\left(\psi + \Sigma \frac{X}{a^{IIIm+1}}\right) + Ia^{IVm}\left(\psi + \Sigma \frac{X}{a^{IVm+1}}\right) + \&c.$$

Si X conſtans fuerit prodibit aequatio

$$y^m = \psi\,(F a^{Im} + G a^{IIm} + H a^{IIIm} + I a^{IVm} + K a^{Vm} + \&c.)$$

$$+ X\left[F\frac{a^{Im}-1}{a^{I}-1} + G\frac{a^{IIm}-1}{a^{II}-1} + H\frac{a^{IIIm}-1}{a^{III}-1} + I\frac{a^{IVm}-1}{a^{IV}-1} + K\frac{a^{Vm}-1}{a^{V}-1} + \&c.\right]$$

Si fuerit $X = 0$ conſtans ψ ſupprimi poterit, & ſimpliciorem aſſequemur aequationem

$$y^m = F a^{Im} + G a^{IIm} + H a^{IIIm} + I a^{IVm} + K a^{Vm} + \&c.$$

quae exhibet terminum generalem ſeriei ipſarum y, hoc eſt ſeriei $y^m + Ay^{m-1} + By^{m-2} + Cy^{m-3} + Dy^{m-4}$ &c. $= 0$, quae ſane nihil aliud eſt quam ſeries recurrens, cuius ſcala relationis eſt $- A - B - C - D - E -$ &c.

10. En igitur ſerierum recurrentium theoriam ad differentialis calculi fundamenta revocatam, & ex genuinis, directiſque principiis depromtam, cum ea prius methodis omnino indirectis, notionibuſque alienis, & e longinquo detortis inniteretur.

Doctrinam hanc praeſtantiſſimam maximique in analyſi univerſa uſus & emolumenti ſelectis aliquot exemplis declarare, ac tyronum captui accommodare operae pretium ducimus.

EXEMPLUM I.

Quaeratur terminus generalis ſeriei recurrentis ſecundi ordinis $1 + 2u^2 + 2u^3 + 6u^4 + 10u^5 + 22u^6 + 42u^7 + 86u^8 +$ &c. *quae oritur ex evolutione fractionis rationalis*

$$\frac{1 - u}{1 - u - 2u},$$

Vocato itaque y'' termino generali ſeriei numericae

1,

1, 0, 2, 2, 6, 10 &c., & y', y terminis binis ipſum y'' proxime antecedentibus, habebitur $y''=y'+2y$, exiſtente nimirum $1+2$ *relationis ſcala*. Iam vero ob $y'=y+\Delta y$, & $y''=y+2\Delta y+\Delta^2 y$, convertitur aequatio $y''=y'+2y$ in hanc $y+2\Delta y+\Delta^2 y=3y+\Delta y$; unde oritur

(A) $y-\frac{1}{2}\Delta y-\frac{1}{2}\Delta^2 y=0$.

Capio iam $\Delta y=p$, factaque multiplicatione per coefficientem indeterminatum a colligo aequalitatem $ap-a\Delta y=0$, hancque addo aequationi (A) poſt ſubſtitutionem ipſius p loco Δy, & Δp loco $\Delta^2 y$, ex quo nanciſcor aequationem

(B) $y+(a-\frac{1}{2})p-a\Delta y-\frac{1}{2}\Delta p=0$.

Sit iam huius aequationis pars $y+(a+\frac{1}{2})p$ aequalis integrali partis alterius $-a\Delta y-\frac{1}{2}dp$ per $-a$ diviſae, hoc eſt fiat $y+(a-\frac{1}{2})p=y+\frac{p}{2a}$, unde ex terminorum comparatione oritur $a-\frac{1}{2}=\frac{1}{2a}$, nimirum $a^2-\frac{a}{2}-\frac{1}{2}=0$; atque inde $a'=1$, $a''=-\frac{1}{2}$. Pono $z=y+(a-\frac{1}{2})p$, & aequatio (B) tranſit in $z-a\Delta z=0$, vel $\Delta z-\frac{z}{a}=0$, quae comparata cum aequatione (§. 3.) $\Delta y+(1-R)y=T$ praebet $T=0$,

$1-R=-\frac{1}{a}$, ſeu $R=\frac{a+1}{a}$, $y=z$. Igitur

$$z^m=R^m\left(A+\Sigma\frac{T}{R^{m+1}}\right)=\left(\frac{a'+1}{a'}\right)^m A=2^m A,$$ & rurſus

$$z^m=\left(\frac{a''+1}{a''}\right)^m B=(-1)^m B.$$

Eſt autem $z^m=y^m+(a'-\frac{1}{2})p=y^m+\frac{1}{2}p$; & iterum $z^m=y^m+(a''-\frac{1}{2})p=y^m-p$. Igitur ſubſtitutis valoribus binis inventis loco z^m oriuntur aequationes duae $2^m A=y^m+\frac{1}{2}p$, $(-1)^m B=y^m-p$, ex quibus eliminata

Ponamus ut ante terminos binos $z \Delta u + M u z = 0$, & orietur $\Delta u + M u = 0$, ſeu $\frac{\Delta u}{u} = -M$: ad hanc aequationem in hac hypotheſi differentiae finitae Δu integrandam aſſumo $u = e^t$, & nanciſcor $u + \Delta u = e^{t + \Delta t}$, & $\Delta u = e^t (e^{\Delta t} - 1)$; ex quo fit $\frac{\Delta u}{u} = e^{\Delta t} - 1 = -M$, ſeu $e^{\Delta t} = 1 - M$, captiſque logarithmis $\Delta t = l(1 - M)$, & conſequenter inſtituta integratione habebimus $t = \Sigma\, l(1 - M)$. Eſt autem, ut ex Analyſi conſtat, aggregatum ex logarithmis plurium numerorum aequale logarithmo producti horum omnium numerorum; ſi igitur generatim per $\pi(1 - M)$ exprimamus productum continuum quantitatum omnium contentarum in formula $1 - M$, orietur $t = l\,\pi(1 - M)$; & propterea $u = e^t = \pi(1 - M)$. Iam praedictis duobus terminis nulleſcentibus ſuperior aequatio mutatur in $u \Delta z + \Delta u \Delta z = N$, unde eruitur $\Delta z = \frac{N}{u + \Delta u}$; & integrando $z = \Sigma \frac{N}{u + \Delta u}$. Sed cum iam invenerimus $u = \pi(1 - M)$, ſi terminus poſt M proxime ſubſequens deſignetur per M', prodit $u + \Delta u = \pi(1 - M')$; ideoque $z = \Sigma \frac{N}{\pi(1-M')}$, & quia $y = z u$, fit

$$y = \pi(1 - M) \Sigma \frac{N}{\pi(1-M')},$$

ſive addita conſtante quacunque A conſequemur

$$y = \pi(1 - M) \left(A + \Sigma \frac{N}{\pi(1-M')}\right).$$

EXEMPLUM.

Sit propoſita aequatio $y + (x + 1) \Delta y + a(2x + 1) = 0$, *quaeraturque valor ſeparatus ipſius* y.

Haec aequatio ad generalem formam $\Delta y + M y = N$ reducta dat $\Delta y + \frac{1}{x+1} y = -\frac{a(2x+1)}{x+1}$. Habemus ergo M =

$M = \frac{1}{x+1}$; $N = -\frac{a(2x+1)}{x+1}$; $1 - M = \frac{x}{x+1}$; quapropter productum omnium valorum formulae $\frac{x}{x+1}$, qui habentur ſi in ea loco x ſucceſſive $x-1, x-2, \ldots 3, 2, 1$, ſubſtituantur, ſcilicet $\frac{x-1}{x} \cdot \frac{x-2}{x-1} \cdot \frac{x-3}{x-2} \cdot \ldots \cdot \frac{1}{2}$, & per $\pi(1-M)$ indicavimus, erit $= \frac{1}{x}$; illud vero per $\pi(1-M')$ expreſſum $= \frac{1}{x+1}$. Hinc hiſce valoribus in aequatione

$y = \pi(1-M)\left(A + \Sigma \frac{N}{\pi(1-M')}\right)$ ſubſtitutis habebitur

$y = \frac{1}{x}\left[A - \Sigma\, a(2x+1)\right] = \frac{A}{x} - 2a\frac{\Sigma x}{x} - a\frac{\Sigma 1}{x}$.

Inventum eſt autem §. 60. $\Sigma x^0 = \Sigma 1 = x$, & $\Sigma x = \frac{1}{2}x^2 - \frac{1}{2}x$ proindeque erit $y = \frac{A}{x} - ax$

3. Sit modo propoſita aequatio $y' = Ry + T$, in qua y' terminum deſignat, qui in ſerie quantitatum y proxime ſequitur y: hinc quoniam $y' = y + \Delta y$, aequatio abit in $\Delta y + (1-R)y = T$. Inſtituta huius aequationis cum praecedenti comparatione prodit $1 - R = M$; $T = N$. Quapropter pro valore quantitatis y ſequens habetur expreſſio

$$y = \pi R\left(A + \Sigma \frac{T}{\pi R'}\right).$$

Si R quantitas conſtans eſt, perſpicuum ſit quantitates πR & $\pi R'$ nihil aliud eſſe quam potentias ipſius R, quarum exponens aequatur numero deſignanti in ſerie ipſarum y locum vel indicem terminorum y & y'. Sit igitur m numerus iſte, vel index loci ab y occupati ita ut y^m idem ſit ac y', & habe-

habebitur aequatio $y^m = R^m \left(A + \Sigma \frac{T}{R^{m+1}}\right)$. Si T conſtans eſt, fit $\Sigma \frac{T}{R^{m+1}} = T \Sigma \frac{1}{R^{m+1}}$, ubi termini per $\frac{1}{R^{m+1}}$ exhibiti progreſſionem geometricam conſtituunt, cuius ſumma protinus invenitur: ſumma haec ab $\frac{1}{R}$ incipiens, vocetur S, ponatur nimirum

$$\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \ldots\ldots + \frac{1}{R^m} = S,$$

& multiplicando per R conſequemur

$$1 + \frac{1}{R} + \frac{1}{R^2} + \ldots\ldots + \frac{1}{R^{m-1}} = SR = S + 1 - \frac{1}{R^m}.$$

Ex hac aequatione elicitur $S = \frac{R^m - 1}{R^m(R-1)}$; atque inde $y^m = R^m \left(A + T.\frac{R^m-1}{R^m(R-1)}\right)$, ſive $y^m = AR^m + T.\frac{R^m-1}{R-1}$. Ut nunc oſtendamus per inventum ipſius y valorem omnimode ſatisfieri conditionibus datae aequationis $y' = Ry + T$, vel $y^{m+1} = Ry^m + T$, non alia re opus erit quam multiplicatione inventae formulae per R, & additione quantitatis T, ex quo colligitur expreſſio $AR^{m+1} + T.\frac{R^{m+1}-R}{R-1} + T$, quae redigitur ad $AR^{m+1} + T.\frac{R^{m+1}-1}{R-1}$, hicque revera eſt valor, quem generalis formula pro termino y^{m+1} ſuppeditat.

4. Tradita iam methodo integrandi aequationem quamcumque differentialem ex differentiis finitis conſtantem, & ſub generali forma $\Delta y + My = N$ comprehenſam, reliquum eſt ut ad integrandas aequationes alias ab hac eadem pendentes progrediamur. Iam vero demonſtravit ALEMBERTUS in Actis Beroli-

Berolinenſis Academiae, aequationes omnes differentiales, e differentiis infiniteſimis conſtantes huiuſce formae

$$\text{(A)} \qquad y + \frac{A\,dy}{dx} + \frac{B\,ddy}{dx^2} + \frac{C\,d^3y}{dx^3} + \text{\&c.} = X.$$

ubi A, B, C, &c. conſtantes ſunt, & X functio eſt quaecunque ipſius x, revocari & converti in aequationem hanc ſimpliciorem $z + \frac{H\,dz}{dx} = V$, ubi H conſtans eſt, & V functio ipſius x; quae ſane aequatio eadem eſt cum ea, quam integrare docuimus ſuppoſitis etiam differentiis finitis. Si igitur Alembertiana methodus aequationibus etiam differentiarum finitarum accommodari poteſt licebit quoque integrare in hac ipſa differentiarum finitarum hypotheſi aequationem quamlibet

$$y + A\Delta y + B\Delta\Delta y + C\Delta^3 y + \text{\&c.} = X,$$

& conſequenter aequationem huius formae

$$y^{\text{I}} + Py^{\text{II}} + Qy^{\text{III}} \ldots + \text{\&c.} = X$$

quae pro generali ſerierum recurrentium formula haberi merito poteſt. Ingenioſiſſima Alembertiana methodus, quae *coefficientium indeterminatorum* methodus appellatur, in hoc eſt poſita: Sumitur ſcilicet

$$\frac{dy}{dx} = p \;\; ; \;\; \frac{dp}{dx} = q \;\; ; \;\; \frac{dq}{dx} = r \;\; ; \;\; \text{\&c.}$$

indeque fiunt aequationes

$$p - \frac{dy}{dx} = 0\,;\;\; q - \frac{dp}{dx} = 0\,;\;\; r - \frac{dq}{dx} = 0\,;\;\; \text{\&c.}$$

Tum aequationes hae ſingulae in coefficientes indeterminatos a, b, c, &c. ducuntur, ut oriatur

$$ap - \frac{ady}{dx} = 0\,;\; bq - \frac{bdp}{dx} = 0\,;\; cr - \frac{cdq}{dx} = 0\,;\; \text{\&c.}$$

Hae vero poſtremae addantur aequationi (A), quae ſi ultra tertias differentias variabilis y non procedat, mutatur factis debitis ſubſtitutionibus in ſequentem

$$(B)\ldots y+(A+a)p+(B+b)q-\frac{ady}{dx}-\frac{bdp}{dx}+\frac{Cdq}{dx}=X.$$

In primo huius aequationis membro ſumamus nunc partem unam $y+(A+a)p+(B+b)q$ ductam in a eſſe multiplam integralis ex parte altera $ady+bdp-Cdq$ reſultantis, vel quod eodem redit eſſe

$$dy+(A+a)dp+(B+b)\,dq=dy+\frac{bdp}{a}-\frac{Cdq}{a};$$

unde ex terminorum ſibi reſpondentium comparatione prodeunt aequalitates $A+a=\frac{b}{a}$; $B+b=-\frac{C}{a}$, ex quibus conſequimur $b=-\frac{C}{a}-B=Aa+a^2$, & $a^3+Aa^2+Ba+C=0$.

Huius porro aequationis $a^3+Aa^2+Ba+C=0$ radices praebent valores tres ipſius a diverſos, qui requiſitis conditionibus aeque ſatisfaciunt. Fiat nunc

$$y+(A+a)p+(B+b)\,q=z,$$

& aequatio inventa (B) mutabitur in

$$z-\frac{adz}{dx}=X,\ \text{ſeu}\ dz-\frac{zdx}{a}=-\frac{Xdx}{a}.$$

Hac comparata cum §. 1. aequatione $dy+yPdx=Zdx$ (ubi ad ambiguitatem vitandam X convertimus in P) habetur

$$y=z;\ P=-\frac{1}{a};\ Z=-\frac{X}{a};\ e^{\int Pdx}=e^{\int-\frac{dx}{a}}=e^{-\frac{x}{a}},$$

ex quo demum colligitur $z=e^{\frac{x}{a}}\int\frac{-X\,dx}{ae^{\frac{x}{a}}}$. Voco nunc a^{I}, a^{II}, a^{III} tres diſtinctos ipſius a valores, & b^{I}, b^{II}, b^{III} valores alios ipſius b prioribus homologos, ac denique Z^{I}, Z^{II}, Z^{III} valores quantitatis variabilis z, quae per vices complectitur a^{I}, a^{II}, a^{III}. Hinc tres ſequentes prodibunt aequationes

$$y+(A+a^{\text{I}})p+(B+b^{\text{I}})q=Z^{\text{I}}$$
$$y+(A+a^{\text{II}})p+(B+b^{\text{II}})q=Z^{\text{II}}$$
$$(1)\quad y+(A+a^{\text{III}})p+(B+b^{\text{III}})q=Z^{\text{III}}$$

Si

Si iam ex tribus hiſce aequationibus eliminantur quantitates p & q, invenitur ipſius y valor ſequenti aequatione exhibitus

$$y = FZ^{I} + GZ^{II} + HZ^{III}$$

ubi F, G, H conſtantes ſunt a quantitatibus A, B, a^{I}, a^{II}, &c. pendentes.

5. Hac methodo probe perpenſa luculenter apparet aequationem quamlibet multo pluribus terminis conflatam, cuiuſmodi eſſet iſthaec

$$y + \frac{A\,dy}{dx} + \frac{B\,ddy}{dx^2} + \frac{C\,d^3y}{dx^3} + \frac{D\,d^4y}{dx^4} + \frac{E\,d^5y}{dx^5} = X$$

aeque reſolvi poſſe, indeque obtineri

$$y = FZ^{I} + GZ^{II} + HZ^{III} + IZ^{IV} + KZ^{V},$$

ubi quantitates Z^{I}, Z^{II}, &c. tales ſunt ipſarum X & x functiones, ut ſit $Z = e^{\frac{x}{a}} \int \frac{-X\,dx}{a e^{\frac{x}{a}}}$, ſubſtitutis pro a radicibus quinque a^{I}, a^{II}, a^{III}, a^{IV}, a^{V} aequationis

$$a^5 + Aa^4 + Ba^3 + Ca^2 + Da + E = 0.$$

Sed quod longe utilius eſt ac praeſtantius methodus eadem ad aequationes quoque ex finitis differentiis conſtantes feliciſſime aptari poteſt.

6. Eſto itaque aequatio differentialis finita quinti ordinis

$$y + A\Delta y + B\Delta^2 y + C\Delta^3 y + D\Delta^4 y + E\Delta^5 y = X$$

& ponatur $\Delta y = p$; $\Delta p = q$; $\Delta q = r$; $\Delta r = s$, ut prodeant aequalitates

$$p - \Delta y = 0;\ q - \Delta p = 0;\ r - \Delta q = 0;\ s - \Delta r = 0$$

hiſque multiplicatis per coefficientes indeterminatos a, b, c, d fiat $ap - a\Delta y = 0$; $bq - b\Delta p = 0$; $cr - c\Delta q = 0$; $ds - d\Delta r = 0$. Aequationi propoſitae poſt ſubſtitutionem valorum aſſumtorum pro differentiis ipſius y addantur aequalitates modo inventae; quo facto eruitur aequatio

$$y + (A + a)p + (B + b)q + (C + c)r + (D + d)s$$
$$- a\Delta y - b\Delta p - c\Delta q - d\Delta r + E\Delta s = X$$

huius aequationis pars

 $y +$

$$y+(A+a)p+(B+b)q+(C+c)r+(D+d)s$$

differentiata aequalis aſſumatur parti alteri

$$a\Delta y+b\Delta p+c\Delta q+d\Delta r-E\Delta s$$

per $-a$ diviſae, ſcilicet fiat

$$\Delta y+(A+a)\Delta p+(B+b)\Delta q+(C+c)\Delta r+(D+d)\Delta s=$$

$$\Delta y+\frac{b\Delta p}{a}+\frac{c\Delta q}{a}+\frac{d\Delta r}{a}-\frac{E\Delta s}{a}$$

unde per terminorum homologorum comparationem inferuntur aequalitates

$$A+a=\frac{b}{a};\ B+b=\frac{c}{a};\ C+c=\frac{d}{a};\ D+d=-\frac{E}{a},$$

ex quibus per conſuetas eliminationes exurgit aequatio quinti gradus

$$a^5+Aa^4+Ba^3+Ca^2+Da+E=0:$$

huius radices praebent quinque ipſius a diſtinctos valores a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}. Iam vero capiatur

$$y+(A+a)p+(B+b)q+(C+c)r+(D+d)s=z$$

ideoque

$$\Delta z=\Delta y+(A+a)\Delta p+(B+b)\Delta q+(C+c)\Delta r+(D+d)\Delta s=$$

$$\Delta y+\frac{b}{a}\Delta p+\frac{c}{a}\Delta q+\frac{d}{a}\Delta r-\frac{E}{a}\Delta s,$$

unde oritur aequatio

$$z-a\Delta z=X,\ \text{hoc eſt}\ \Delta z-\frac{z}{a}=-\frac{X}{a}$$

eiuſdem ſcilicet formae cum praecedenti $\Delta y+My=N$ (§. 2.) ubi ex terminorum homologorum comparatione deducitur

$$M=-\frac{1}{a};\ N=-\frac{X}{a},\ \&\ \text{conſequenter}\ 1-M=\frac{1+a}{a}.$$

Quapropter oritur $z=\pi\left(\frac{1+a}{a}\right)\left(\text{Conſt.}+\Sigma\frac{-X}{a.\pi\left(\frac{1+a}{a}\right)}\right);$

Et quoniam a conſtans eſt, invenitur

$$z^m = \left(\frac{1+a}{a}\right)^m \left(\text{Conſt.} - \Sigma\, \frac{X a^m}{(1+a)^{m+1}}\right)$$

denotante m ut antea indicem loci a termino z occupati in ſerie quantitatum z. Si X conſtans eſt tunc progreſſionis geometricae per $\Sigma\, \frac{a^m}{(1+a)^{m+1}}$ expreſſae accepta ſumma aſſequimur $z^m = \left(\frac{1+a}{a}\right)^m \left(\text{Conſt.} - X\, \frac{(1+a)^m - a^m}{(1+a)^m}\right)$.

Quum autem a quinque valoribus diverſis $a^{\text{I}}, a^{\text{II}}, a^{\text{III}}, a^{\text{IV}}, a^{\text{V}}$ praeditus ſit, ſi valores iſti in formula inventa ſubrogantur per vices, totidem exſurgunt ipſius z^m valores, qui omnes aeque ſatisfacient: his vero per $Z^{\text{I}}, Z^{\text{II}}, Z^{\text{III}}, Z^{\text{IV}}, Z^{\text{V}}$ deſignatis quinque ſequentes naniſcimur aequationes

$$y + (A + a^{\text{I}})p + (B + b^{\text{I}})q + (C + c^{\text{I}})r + (D + d^{\text{I}})s = Z^{\text{I}}$$
$$y + (A + a^{\text{II}})p + (B + b^{\text{II}})q + (C + c^{\text{II}})r + (D + d^{\text{II}})s = Z^{\text{II}}$$
$$y + (A + a^{\text{III}})p + (B + b^{\text{III}})q + (C + c^{\text{III}})r + (D + d^{\text{III}})s = Z^{\text{III}}$$
$$y + (A + a^{\text{IV}})p + (B + b^{\text{IV}})q + (C + c^{\text{IV}})r + (D + d^{\text{IV}})s = Z^{\text{IV}}$$
$$y + (A + a^{\text{V}})p + (B + b^{\text{V}})q + (C + c^{\text{V}})r + (D + d^{\text{V}})s = Z^{\text{V}}.$$

Tum ex hiſce aequationibus per notas regulas eliminantur quantitates p, q, r, s, quo facto ad aequationem pervenitur huius formae $y = FZ^{\text{I}} + GZ^{\text{II}} + HZ^{\text{III}} + IZ^{\text{IV}} + KZ^{\text{V}}$, in qua F, G, H, I, K conſtantes ſunt a cognitis quantitatibus aequationum earundem pendentes.

7. Propoſita demum ſit aequatio

$$y^{\text{I}} + Ay^{\text{II}} + By^{\text{III}} + Cy^{\text{IV}} \ldots + \&c. = X$$

ubi $y^{\text{I}}, y^{\text{II}}, y^{\text{III}}$, &c. terminos deſignant ſe proxime ſequentes in ſerie quantitatum y. Perſpicuum imprimis eſt haberi

$$y^{\text{II}} = y^{\text{I}} + \Delta y$$
$$y^{\text{III}} = y^{\text{II}} + \Delta y^{\text{II}} = y^{\text{I}} + \Delta y^{\text{I}} + \Delta(y^{\text{I}} + \Delta y^{\text{I}}) = y^{\text{I}} + 2\Delta y^{\text{I}} + \Delta^2 y^{\text{I}}$$
$$y^{\text{IV}} = y^{\text{III}} + \Delta y^{\text{III}} = y^{\text{I}} + 2\Delta y^{\text{I}} + \Delta^2 y^{\text{I}} + \Delta(y^{\text{I}} + 2\Delta y^{\text{I}} + \Delta^2 y^{\text{I}})$$
$$= y^{\text{I}} + 3\Delta y^{\text{I}} + 3\Delta^2 y^{\text{I}} + \Delta^3 y^{\text{I}},$$

ſicque porro de ceteris. Ex quo apparet aequationem propoſitam ad formam earum, quas ante examinavimus, commode

de revocari. Verum ut facilius ac promptius aequatio ipſa ad ſeries recurrentes accommodari queat praeſtabit terminos $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$, &c. ordine inverſo ſpectare, ut ſcilicet habeatur $y^{\mathrm{II}} + \Delta y^{\mathrm{II}} = y^{\mathrm{I}}$; $y^{\mathrm{III}} + \Delta y^{\mathrm{III}} = y^{\mathrm{II}}$; $y^{\mathrm{IV}} + \Delta y^{\mathrm{IV}} = y^{\mathrm{III}}$, &c. & exponentes I, II, III, &c. deſignent cuiuslibet termini diſtantiam a poſtremo y^{I}. Ponatur $y^{\mathrm{II}} = p^{\mathrm{I}}$, eritque $y^{\mathrm{III}} = p^{\mathrm{II}}$: fiat iterum $p^{\mathrm{II}} = q^{\mathrm{I}}$, proindeque $p^{\mathrm{III}} = q^{\mathrm{II}}$; ſit rurſus $q^{\mathrm{II}} = r^{\mathrm{I}}$, ideoque $q^{\mathrm{III}} = r^{\mathrm{II}} = s^{\mathrm{I}}$. Ex quo oriuntur aequalitates $y^{\mathrm{II}} = p^{\mathrm{I}}$; $y^{\mathrm{III}} = q^{\mathrm{I}}$; $y^{\mathrm{IV}} = r^{\mathrm{I}}$; $y^{\mathrm{V}} = s^{\mathrm{I}}$; $y^{\mathrm{VI}} = s^{\mathrm{II}}$: quibus valoribus in aequatione propoſita ſubrogatis ea abit in hanc (A) .... $y^{\mathrm{I}} + \mathrm{A}p^{\mathrm{I}} + \mathrm{B}q^{\mathrm{I}} + \mathrm{C}r^{\mathrm{I}} + \mathrm{D}s^{\mathrm{I}} + \mathrm{E}s^{\mathrm{II}} = \mathrm{X}$. Iam vero aequationibus $p^{\mathrm{I}} - y^{\mathrm{II}} = 0$; $q^{\mathrm{I}} - p^{\mathrm{II}} = 0$; $r^{\mathrm{I}} - q^{\mathrm{II}} = 0$; $s^{\mathrm{I}} - r^{\mathrm{II}} = 0$ per indeterminatos coefficientes a, b, c, d ſingillatim multiplicatis, hiſque de more additis aequationi (A) invenitur ſequens

$$\left.\begin{matrix} y^{\mathrm{I}} + (\mathrm{A}+a)p^{\mathrm{I}} + (\mathrm{B}+b)q^{\mathrm{I}} + (\mathrm{C}+c)r^{\mathrm{I}} + (\mathrm{D}+d)s^{\mathrm{I}} \\ - ay^{\mathrm{II}} - bp^{\mathrm{II}} - cq^{\mathrm{II}} - dr^{\mathrm{II}} + \mathrm{E}s^{\mathrm{II}} \end{matrix}\right\} = \mathrm{X}.$$

In prima huius aequationis parte

$$y^{\mathrm{I}} + (\mathrm{A}+a)p^{\mathrm{I}} + (\mathrm{B}+b)q^{\mathrm{I}} + (\mathrm{C}+c)r^{\mathrm{I}} + (\mathrm{D}+d)s^{\mathrm{I}}$$

variantes termini tranſeant in proxime antecedentes ita ut fiat

$$y^{\mathrm{II}} + (\mathrm{A}+a)p^{\mathrm{II}} + (\mathrm{B}+b)q^{\mathrm{II}} + (\mathrm{C}+c)r^{\mathrm{II}} + (\mathrm{D}+d)s^{\mathrm{II}}$$

haecque quantitas ponatur aequalis parti alteri per $-a$ diviſae

ſeu $$y^{\mathrm{II}} + \frac{b}{a}p^{\mathrm{II}} + \frac{c}{a}q^{\mathrm{II}} + \frac{d}{a}r^{\mathrm{II}} - \frac{\mathrm{E}}{a}s^{\mathrm{II}}$$

tum aequatis coefficientibus homologis habentur aequationes

$$\mathrm{A}+a = \frac{b}{a};\ \mathrm{B}+b = \frac{c}{a};\ \mathrm{C}+c = \frac{d}{a};\ \mathrm{D}+d = -\frac{\mathrm{E}}{a},$$

ex quibus eruitur ut ante quinti gradus aequatio

$$a^5 + \mathrm{A}a^4 + \mathrm{B}a^3 + \mathrm{C}a^2 + \mathrm{D}a + \mathrm{E} = 0$$

cuius radices deſignantur per $a^{\mathrm{I}}, a^{\mathrm{II}}, a^{\mathrm{III}}, a^{\mathrm{IV}}, a^{\mathrm{V}}$. Quocirca accepto $z^{\mathrm{I}} = y^{\mathrm{I}} + (\mathrm{A}+a)p^{\mathrm{I}} + (\mathrm{B}+b)q^{\mathrm{I}} + (\mathrm{C}+c)r^{\mathrm{I}} + (\mathrm{D}+d)s^{\mathrm{I}}$, aequatio propoſita tranſibit in hanc $z^{\mathrm{I}} - az^{\mathrm{II}} = \mathrm{X}$, quae (ob $z^{\mathrm{I}} = z^{\mathrm{II}} + \Delta z^{\mathrm{II}}$) evadet $\Delta z^{\mathrm{II}} + (1-a)z^{\mathrm{II}} = \mathrm{X}$. Hac autem comparata cum aequatione §. 3. oritur $y = z^{\mathrm{II}}$,

R

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$R = a$, $T = X$; fitque propterea $z^m = a^m\left(\text{Conſt.} + \Sigma \frac{X}{a^{m+1}}\right)$ ubi m locum denotat a termino z^m in ſerie ipſarum z occupatum. Quum autem loco ipſius a ſubſtitui poſſint per vices radices ſingulae a^{I}, a^{II}, &c. inde conſequitur, quantitatis z^m valores quinque diverſos per Z^{I}, Z^{II} Z^{III}, Z^{IV}, Z^{V} expreſſos oriri, proindeque aequationes quinque ſequentes produci

$$y^m + (A + a^{\mathrm{I}})p^m + (B + b^{\mathrm{I}})q^m + (C + c^{\mathrm{I}})r^m + (D + d^{\mathrm{I}})s^m = Z^{\mathrm{I}}$$
$$y^m + (A + a^{\mathrm{II}})p^m + (B + b^{\mathrm{II}})q^m + (C + c^{\mathrm{II}})r^m + (D + d^{\mathrm{II}})s^m = Z^{\mathrm{II}}$$
$$y^m + (A + a^{\mathrm{III}})p^m + (B + b^{\mathrm{III}})q^m + (C + c^{\mathrm{III}})r^m + (D + d^{\mathrm{III}})s^m = Z^{\mathrm{III}}$$
$$y^m + (A + a^{\mathrm{IV}})p^m + (B + b^{\mathrm{IV}})q^m + (C + c^{\mathrm{IV}})r^m + (D + d^{\mathrm{IV}})s^m = Z^{\mathrm{IV}}$$
$$y^m + (A + a^{\mathrm{V}})p^m + (B + b^{\mathrm{V}})q^m + (C + c^{\mathrm{V}})r^m + (D + d^{\mathrm{V}})s^m = Z^{\mathrm{V}}.$$

Eliminatis porro quantitatibus p^m, q^m, r^m, s^m ex hiſce aequationibus pervenitur ad formulam

$$y^m = FZ^{\mathrm{I}} + GZ^{\mathrm{II}} + HZ^{\mathrm{III}} + IZ^{\mathrm{IV}} + KZ^{\mathrm{V}},$$

in qua F, G, H, &c. conſtantes ſunt, quae per comparationem totidem terminorum ſeriei ipſarum y determinari debent.

8. Si X conſtans eſt ſumma (§. 7.) per $\Sigma \frac{X}{a^{m+1}}$ expreſſa evadet $= X \frac{a^m - 1}{a^m(a-1)}$; dictaque L conſtanti integrationi addenda nanciſcimur tandem $Z = La^m + X \frac{a^m - 1}{a - 1}$, ex qua protinus habentur valores Z^{I}, Z^{II}, Z^{III}, &c. ſubſtitutis dumtaxat loco ipſius a radicibus inventis a^{I}, a^{II}, a^{III}, &c.

9. Ex hactenus dictis hoc ſtatim deducitur generale theorema:

Propoſita aequatione $y^m + Ay^{m-1} + By^{m-2} + Cy^{m-3} \ldots + \&c. = X$, *ubi ipſius* y *exponentes deſignant loca ab* y *occupata; inventiſque radicibus* $a^{\mathrm{I}}, a^{\mathrm{II}}, a^{\mathrm{III}}$ *&c. aequationis*

$$a^n + Aa^{n-1} + Ba^{n-2} + \&c. = 0,$$

habebitur generatim

$$y^m = \mathrm{F}a^{\mathrm{I}m}\left(\psi + \Sigma \frac{\mathrm{X}}{a^{\mathrm{I}m+1}}\right) + \mathrm{G}a^{\mathrm{II}m}\left(\psi + \Sigma \frac{\mathrm{X}}{a^{\mathrm{II}m+1}}\right)$$

$$+ \mathrm{H}a^{\mathrm{III}m}\left(\psi + \Sigma \frac{\mathrm{X}}{a^{\mathrm{III}m+1}}\right) + \mathrm{I}a^{\mathrm{IV}m}\left(\psi + \Sigma \frac{\mathrm{X}}{a^{\mathrm{IV}m+1}}\right) + \&c.$$

Si X conſtans fuerit prodibit aequatio

$$y^m = \psi\,(\mathrm{F}a^{\mathrm{I}m} + \mathrm{G}a^{\mathrm{II}m} + \mathrm{H}a^{\mathrm{III}m} + \mathrm{I}a^{\mathrm{IV}m} + \mathrm{K}a^{\mathrm{V}m} + \&c.)$$

$$+ \mathrm{X}\left[\mathrm{F}\frac{a^{\mathrm{I}m}-1}{a^{\mathrm{I}}-1} + \mathrm{G}\frac{a^{\mathrm{II}m}-1}{a^{\mathrm{II}}-1} + \mathrm{H}\frac{a^{\mathrm{III}m}-1}{a^{\mathrm{III}}-1} + \mathrm{I}\frac{a^{\mathrm{IV}m}-1}{a^{\mathrm{IV}}-1} + \right.$$

$$\left. \mathrm{K}\frac{a^{\mathrm{V}m}-1}{a^{\mathrm{V}}-1} + \&c.\right]$$

Si fuerit $\mathrm{X} = 0$ conſtans ψ ſupprimi poterit, & ſimpliciorem aſſequemur aequationem

$$y^m = \mathrm{F}a^{\mathrm{I}m} + \mathrm{G}a^{\mathrm{II}m} + \mathrm{H}a^{\mathrm{III}m} + \mathrm{I}a^{\mathrm{IV}m} + \mathrm{K}a^{\mathrm{V}m} + \&c.$$

quae exhibet terminum generalem ſeriei ipſarum y, hoc eſt ſeriei $y^m + \mathrm{A}y^{m-1} + \mathrm{B}y^{m-2} + \mathrm{C}y^{m-3} + \mathrm{D}y^{m-4}\ \&c. = 0$, quae ſane nihil aliud eſt quam ſeries recurrens, cuius ſcala relationis eſt $-\mathrm{A} - \mathrm{B} - \mathrm{C} - \mathrm{D} - \mathrm{E} - \&c.$

10. En igitur ſerierum recurrentium theoriam ad differentialis calculi fundamenta revocatam, & ex genuinis, directiſque principiis depromtam, cum ea prius methodis omnino indirectis, notionibuſque alienis, & e longinquo detortis inniteretur.

Doctrinam hanc praeſtantiſſimam maximique in analyſi univerſa uſus & emolumenti ſelectis aliquot exemplis declarare, ac tyronum captui accommodare operae pretium ducimus.

EXEMPLUM I.

Quaeratur terminus generalis ſeriei recurrentis ſecundi ordinis
$1 + 2u^2 + 2u^3 + 6u^4 + 10u^5 + 22u^6 + 42u^7 + 86u^8 + \&c.$
quae oritur ex evolutione fractionis rationalis

$$\frac{1-u}{1-u-2u},$$

Vocato itaque y^m termino generali ſeriei numericae

1,

1, 0, 2, 2, 6, 10 &c., & y', y terminis binis ipſum y'' proxime antecedentibus, habebitur $y''=y'+2y$, exiſtente nimirum $1+2$ *relationis ſcala*. Iam vero ob $y'=y+\Delta y$, & $y''=y+2\Delta y+\Delta^2 y$, convertitur aequatio $y''=y'+2y$ in hanc $y+2\Delta y+\Delta^2 y=3y+\Delta y$; unde oritur

(A) $$y-\tfrac{1}{2}\Delta y-\tfrac{1}{2}\Delta^2 y=0.$$

Capio iam $\Delta y=p$, factaque multiplicatione per coefficientem indeterminatum a colligo aequalitatem $ap-a\Delta y=0$, hancque addo aequationi (A) poſt ſubſtitutionem ipſius p loco Δy, & Δp loco $\Delta^2 y$, ex quo nanciſcor aequationem

(B) $$y+(a-\tfrac{1}{2})p-a\Delta y-\tfrac{1}{2}\Delta p=0.$$

Sit iam huius aequationis pars $y+(a+\frac{1}{2})p$ aequalis integrali partis alterius $-a\Delta y-\frac{1}{2}dp$ per $-a$ diviſae, hoc eſt fiat $y+(a-\frac{1}{2})p=y+\frac{p}{2a}$, unde ex terminorum comparatione oritur $a-\frac{1}{2}=\frac{1}{2a}$, nimirum $a^2-\frac{a}{2}-\frac{1}{2}=0$; atque inde $a'=1$, $a''=-\frac{1}{2}$. Pono $z=y+(a-\frac{1}{2})p$, & aequatio (B) tranſit in $z-a\Delta z=0$, vel $\Delta z-\frac{z}{a}=0$, quae comparata cum aequatione (§. 3.) $\Delta y+(1-R)y=T$ praebet $T=0$,

$$1-R=-\frac{1}{a},\ \text{ſeu}\ R=\frac{a+1}{a},\ y=z.\ \text{Igitur}$$

$$z^m=R^m\left(A+\Sigma\frac{T}{R^{m+1}}\right)=\left(\frac{a'+1}{a'}\right)^m A=2^m A,\ \text{\& rurſus}$$

$$z^m=\left(\frac{a''+1}{a''}\right)^m B=(-1)^m B.$$

Eſt autem $z^m=y^m+(a'-\frac{1}{2})p=y^m+\frac{1}{2}p$; & iterum $z^m=y^m+(a''-\frac{1}{2})p=y^m-p$. Igitur ſubſtitutis valoribus binis inventis loco z^m oriuntur aequationes duae $2^m A=y^m+\frac{1}{2}p$, $(-1)^m B=y^m-p$, ex quibus eliminata

 p

p infertur aequatio $y^m = \frac{2^{m+1} A + (-1)^m B}{3}$. Determinantur porro conſtantes binae arbitrariae A, B ex eo, quod in hypotheſi $m = 0$ fit $y = 1$, & in hypotheſi $m = 1$ fit $y = 0$, ex quo nanciſcimur aequationes duas

$1 = \frac{2A + B}{3}$; $0 = \frac{4A - B}{3}$, quae praebent $A = \frac{1}{2}$, $B = 2$.

Quocirca erit demum terminus generalis quaeſitus

$$\frac{2^m + 2(-1)^m}{3} u^m$$

Quoniam problemata quae ad Analyſim Sortis ſeu Calculum probabilitatis, atque Aleae Ludos referuntur, ex hac aequationum differentiarum finitarum theoria maximum lumen mutuantur, & ſolutiones recipiunt directas; omniumque ſimpliciſſimas, bina idcirco ad rei illuſtrationem proponemus exempla ex aleae ludis depromta.

EXEMPLUM II.

A *certat cum* B *ludo illo, quem Galli dicunt* à croix ou pile, *nummo ſcilicet in altum iacto, cuius frons una ſignatur* croix, *altera* pile: *ludi conditiones ſunt, ut ſi* A *primo iactu edat* croix, *accipiat a* B *aureos duos; ſi* A *ſecundo tantum iactu edat* croix, *accipiat a* B *aureos quatuor; ſi tertio tantum, aureos octo; ſi iactu ſolum* x.^mo^, *aureos* 2^x. *Quaeritur ſpes, ſeu exſpectatio ipſius* A, *hoc eſt quid* A *ante ludum ſolvere debeat ipſi* B, *ut aequa ſit utriuſque ſors.*

Dicatur z_x aureorum numerus ab A in anteceſſum ſolvendus, exiſtente, ut aſſumimus iactuum numero x. Si x augeatur unitate, ita ut pecunia in anteceſſum ab A ſolvenda, ſeu exſpectatio ipſius A exprimatur per z_{x+1}, manifeſtum eſt pecuniam z_x auctum iri aureorum numero 2^{x+1} ducto in probabilitatem $\frac{1}{2^{x+1}}$ eundem lucrandi $\overline{x+1}^{mo}$ iactu. Igitur

habe-

habebimus aequationem $y_{x+1} = y_x + \frac{2^{x+1}}{2^{x+1}} = y_x + 1$, ad quam integrandam reſumo formulam $y' = Ry + T$, & ex terminorum collatione aſſequor $y' = y_{x+1}$; $y = y_x$; $R = 1$; $T = 1$. Quapropter quaeſitum integrale invenietur

$$y = y_x = \pi R\left(A + \Sigma \frac{T}{\pi R}\right) = A + \Sigma . 1 = A + x.$$

Conſtans arbitraria A determinatur ex eo, quod poſito $x = 1$, habetur evidenter exſpectatio $y_x = 1$, proindeque $A + 1 = 1$, ſive $A = 0$. Igitur $y_x = x$, nimirum aureorum numerus ab A ſolvendus $= x$. Q. E. D.

In hoc percelebri & famigerato Problemate certatim elaborarunt Principes huiuſce aetatis Geometrae Dan. Bernoullius, La Fontaine, D'Alembert, La Place, qui ingenioſiſſimis hinc derivatis ſpeculationibus Probabilitatum mirifice Calculum illuſtrarunt.

EXEMPLUM III.

Ex nummorum vel lapillorum acervo, quorum numerus eſt x, caſu arripio multitudinem eorum ignotam: quaero quanta ſit probabilitas, arreptam multitudinem fore parem, aut imparem.

Voco y ſummam caſuum, quibus arrepta multitudo par eſſe poteſt, & z ſummam caſuum, quibus eſſe poteſt impar. Pono augeri unitate acervum x, tuncque y' exhibebit ſummam caſuum parium, eritque $y' = y + z$, eo quod quivis caſus impar cum novo numero compoſitus parem caſum inducit; Praeterea z' indicabit ſummam caſuum imparium, & quia caſus quilibet par cum addito numero coniunctus impar evadit, ipſaque unitas ſuperaddita impar eſt, fiet $z' = z + y + 1$. Iam vero prima aequatio nihil aliud eſt quam $\Delta y = z$, ſecunda nihil aliud quam $\Delta z = y + 1$. Igitur $\Delta^2 y = y + 1$, ſeu $y'' - 2y' + y = y + 1$, & conſequenter $y'' = 2y' + 1$, ſeu quod eodem redit $y' = 2y + 1$. Ex comparatione aequationis huius cum generali $y' = Ry + T$, invenitur $R = 2$, $T = 1$,

 quo-

quocirca habebitur $y = \mathrm{A}\,\mathrm{R}^x + \mathrm{T}\,\frac{\mathrm{R}^x - 1}{\mathrm{R} - 1} = \mathrm{A}\,2^x + 2^x - 1 =$ $(\mathrm{A} + 1)\,2^x - 1$.

Sed facto $x = 1$ fit $y = 0$. Igitur $2\mathrm{A} = 1 - 2$, & $\mathrm{A} = -\frac{1}{2}$; proindeque I.° $y = 2^{x-1} - 1$. Praeterea
II.° $z = \Delta y = y' - y = 2^x - 1 - 2^{x-1} + 1 = 2^x(1 - \frac{1}{2}) = 2^{x-1}$.
Quapropter eventuum omnium probabilium ſumma erit $y + z = 2^{x-1} - 1 + 2^{x-1} = 2^x - 1$, atque inde invenitur probabilitas pro caſibus paribus $\frac{y}{y+z} = \frac{2^{x-1} - 1}{2^x - 1}$, pro imparibus $\frac{z}{y+z} = \frac{2^{x-1}}{2^x - 1}$ Q. E. D.

Examen, quod huius Problematis ſuſcepit **Mairanus** in Hiſtoria Pariſienſis Scientiarum Academiae ad ann. 1728, ſubtile quidem & ingenioſum eſt, ſed in accurata Problematis aeſtimatione Vir alias doctus & acutus incaute offendit.

11. Praeter expoſitam methodum revocandi ſerierum recurrentium doctrinam ad integrationem aequationum linearium finitis differentiis conſtantium, ſimpliciorem aliam multoque elegantiorem protulit ſagaciſſimus idem Geometra De la Grange in Actis Berolinenſis Academiae pro an. 1775; quae ſane methodus, quum omnium longe commodiſſima ſit, atque expeditiſſima in iis praeſertim exemplis, in quibus radices aequationis coefficientium indeterminatorum inveniuntur aequales, peropportunum cenſeo eam hic breviter proponere.

Eſto igitur ſeries recurrens generica

$y_0, y_1, y_2, y_3, \ldots y_x, y_{x+1}, y_{x+2}, \ldots y_{x+r}$,

cuius terminus generalis eſt y_x. Suppono aequationem relationis inter $r + 1$ terminos ſucceſſivos eſſe hanc linearem

(A) $\mathrm{A}y_x + \mathrm{B}y_{x+1} + \mathrm{C}y_{x+2} \ldots \mathrm{K}y_{x+r-1} + y_{x+r} = 0$;

ita ut ſeries recurrens ſit ordinis r, & A, B, C . . . K ſint coefficientes quicunque conſtantes. Eo iam res redit, ut aequatio haec differentialis finita linearis integretur, ſeu, quod idem eſt, ut inveniatur analyticus valor termini generalis y_x in ſerie propoſita.

Ad

Ad hoc conſequendum ſumo $y_x = a z^x$, ubi a, z conſtantes ſunt indeterminatae, Erit igitur $y_{x+1} = a z^{x+1}$; $y_{x+2} = a z^{x+2}$; &c. ex quo ſubſtitutionibus peractis aequatio convertitur in

$$A a z^x + B a z^{x+1} + C a z^{x+2} \ldots + K a z^{x+t-1} + a z^{x+t} = 0$$

ſeu (facta per $a z^x$ diviſione) in

$$\text{(B)} \qquad A + Bz + Cz^2 \;.\;.\;. + K z^{t-1} + z^t = 0$$

Ex hac aequatione manifeſtum eſt I. coefficientem a, qui in ipſa deſideratur arbitrarium eſſe; II. tot haberi diſtinctos ipſius z valores, quot ſunt unitates in indice t. Valores hoſce z, ſeu aequationis radices deſigno per α, β, γ &c.; ſumptiſque pariter diverſis coefficientibus arbitrariis a, b, c &c., totidem oriuntur diſtincti valores ipſius y_x, nimirum $a\alpha^x$, $b\beta^x$, $c\gamma^x$, &c., qui ſane valores, tum ſinguli ſeorſim, tum omnes ſimul aequationi lineari (A) omnino ſatisfaciunt. Habetur itaque in univerſum

$$y_x = a\alpha^x + b\beta^x + c\gamma^x + \text{\&c.}$$

Et quoniam valor hic quantitatis y_x conſtantes arbitrarias a, b, c, &c. numero t complectitur, erit is propterea integrale completum aequationis finitae differentialis (A) ad ordinem t.^{um} pertinentis. Valores autem harum conſtantium a, b, c &c. inveniuntur tribuendo variabili x ſucceſſivos valores 0, 1, 2, 3, 4 &c. & comparando aſſumtam termini generalis y_x expreſſionem in hiſce ipſius x hypotheſibus cum terminis primis ſeriei, qui ſemper noti eſſe debent; ex quibus comparationibus totidem exſurgunt aequationes, quot ſunt conſtantes a, b, c &c., quae hoc modo per functiones y_0, y_1, y_2 &c., & per radices α, β, γ, &c. invenientur expreſſae.

12. In hypotheſi $t = 1$ unica eſſe debet radix aequationis relationis (B); & terminus generalis pro hac hypotheſi fit $y_x = a\alpha^x$; unde ſumto $x = 0$ prodit $y_0 = a\alpha^0 = a$; proindeque $y_x = y_0 \alpha^x$.

In hypotheſi $t = 2$ ob binas aequationis radices α, β evadit $y_x = a\alpha^x + b\beta^x$. Quocirca ſumto $x = 0$, & $x = 1$ prodeunt aequationes duae $y_0 = a + b$; $y_1 = a\alpha + b\beta$, ex quibus indetermi-

terminatarum a, b valores obtinentur huiuſmodi

$$a=\frac{-\beta y_0+y_1}{\alpha-\beta};\quad b=\frac{-\alpha y_0+y_1}{\beta-\alpha}.$$

Sit $r=3$, cui hypotheſi reſpondet generalis terminus $y_x=a\alpha^x+b\beta^x+c\gamma^x$. Captis ſucceſſive $x=0$, $x=1$, $x=2$ habentur tres aequationes

$$y_0=a+b+c;\ y_1=a\alpha+b\beta+c\gamma;\ y_2=a\alpha^2+b\beta^2+c\gamma^2,$$

ex quibus per notas algebrae regulas tandem deducitur

$$a=\frac{\beta\gamma y_0-(\beta+\gamma)y_1+y_2}{(\alpha-\beta)(\alpha-\gamma)};\quad b=\frac{\alpha\gamma y_0-(\alpha+\gamma)y_1+y_2}{(\beta-\alpha)(\beta-\gamma)}$$

$$c=\frac{\alpha\beta y_0-(\alpha+\beta)y_1+y_2}{(\gamma-\alpha)(\gamma-\beta)}$$

Pari modo ſi ſit $r=4$, voceturque δ radix quarta aequationis relationis (B), & f conſtans arbitraria reſpondens in termino generali

$y_x=a\alpha^x+b\beta^x+c\gamma^x+f\delta^x$, capto ſucceſſive $x=0$, $x=1$, $x=2$, $x=3$, quatuor nanciſcimur aequationes

$$\begin{aligned} y_0&=a+b+c+f\\ y_1&=a\alpha+b\beta+c\gamma+f\delta\\ y_2&=a\alpha^2+b\beta^2+c\gamma^2+f\delta^2\\ y_3&=a\alpha^3+b\beta^3+c\gamma^3+f\delta^3 \end{aligned}$$

Ex hiſce porro conſtantes arbitrariae a, b, c, f inveniuntur ut ſequitur.

$$a=\frac{-\beta\gamma\delta y_0+(\beta\gamma+\beta\delta+\gamma\delta)y_1-(\beta+\gamma+\delta)y_2+y_3}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}$$

$$b=\frac{-\alpha\gamma\delta y_0+(\alpha\gamma+\alpha\delta+\gamma\delta)y_1-(\alpha+\gamma+\delta)y_2+y_3}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}$$

$$c=\frac{-\alpha\beta\delta y_0+(\alpha\delta+\beta\delta+\alpha\beta)y_1-(\alpha+\beta+\delta)y_2+y_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}$$

$$f=\frac{-\alpha\beta\gamma y_0+(\alpha\beta+\alpha\gamma+\beta\gamma)y_1-(\alpha+\beta+\gamma)y_2+y_3}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

13. Hinc eliciemus canonem pro determinandis valoribus con-

conſtantium arbitrariarum a, b, c, f, &c. qui ſane eſt huiuſmodi: ſit

(C) $z^t + pz^{t-1} + qz^{t-2} \ldots + fz^2 + gz + K = 0$

aequatio radicum inaequalium α, β, γ, δ, ... φ. Ea aequipollet huic alteri

$(z-\alpha)(z-\beta)(z-\gamma)(z-\delta) \ldots (z-\varphi) = 0$,

ac poſtremus terminus k aequatur producto earundem radicum negative ſumtarum, ſive eſt $k = (-\alpha)(-\beta) \ldots (-\varphi)$; atque hoc productum affirmativum erit, ſi radicum numerus ſit par; negativum ſi impar. Diviſo autem eodem producto per unumquemlibet factorum $-\alpha$, $-\beta$ &c. quotiens contra prodibit affirmativus, ſi radicum numerus impar fuerit; negativus, ſi par. Iam vero conſideratis termini y_0 coefficientibus in valore ipſius a pro variis indicis t hypotheſibus, ſequenti ſchemate res exhibetur

Indices	Coefficientes termini y_0
$t=1$	1
$t=2$	$-\beta$
$t=3$	$\beta\gamma$
$t=4$	$-\beta\gamma\delta$
&c.	&c.

Propterea coefficiens ipſius y_0 in valore arbitrariae conſtantis a erit in univerſum $\frac{k}{-\alpha}$; & ob ſimilitudinem coefficientis ipſius y_0 in valoribus ceteris conſtantium reliquarum b, c, f &c. coefficiens ipſe in valore ipſius b exprimetur per $\frac{k}{-\beta}$; in valore ipſius c per $\frac{k}{-\gamma}$; nimirum aequabitur producto radicum omnium aequationis (C) negative ſumtarum demta illa, quae per coefficientem inveniendum multiplicatur.

14. Exempla ſuperius adducta regulam commonſtrant, quam coefficientes terminorum y_1, y_2, y_3 &c. conſtanter ſervant. Et primo quidem ſigna alternatim procedant, ita ut ſi

coef-

coefficiens termini y_0 affirmativus ſit, negativus evadat in y_1, inde affirmativus redeat in y_2 &c.; & contrarium evenit, ubi coefficiens termini y_0 negativus exiſtat. Praeterea poſito v. g. quod coefficiens termini y_0 ſit productum trium factorum $-\beta$, $-\gamma$, $-\delta$ in valore conſtantis a, erit coefficiens termini y_1 aggregatum omnium binariorum, quae ex tribus ipſis factoribus effici poſſunt, & coefficiens termini y_2 erit aggregatum factorum eorundem $-\beta, -\gamma, -\delta$; denique erit unitas coefficiens termini poſtremi y_3. Hoc modo ſuppoſitis in aequatione (C) radicibus quinque inaequalibus $\alpha, \beta, \gamma, \delta, \varepsilon$ erit termini y_0 coefficiens $= \beta\gamma\delta\varepsilon$; termini y_1 coefficiens $= -(\beta\gamma\delta + \beta\gamma\varepsilon + \beta\delta\varepsilon + \gamma\delta\varepsilon)$; termini y_2 coefficiens $= \beta\gamma + \beta\delta + \beta\varepsilon + \gamma\delta + \gamma\varepsilon + \delta\varepsilon$; termini y_3 coefficiens $= -(\beta + \gamma + \delta + \varepsilon)$; denique termini ultimi y_4 coefficiens $= 1$. Eodem modo determinabimus coefficientes terminorum y_0, y_1, y_2 &c. pro valoribus conſtantium reliquarum b, c, f &c. Quod porro ſpectat ad valorum a, b, c &c. denominatores, ſtatim apparebit denominatorem ipſius a obtineri ſi in aequatione (C) vel $(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)\ldots(z-\varphi) = 0$ ſupprimatur factor $z-\alpha$, ſcribaturque $z=\alpha$, ita ut prodeat

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\ldots(\alpha-\varphi),$$

qui erit denominator quaeſitus. Ita pro denominatore ipſius b dividendum erit productum idem integrum per $z-\beta$, & in quotiente ſcribere oportebit $z=\beta$, quo fiet ipſius b denominator $= (\beta-\alpha)(\beta-\gamma)(\beta-\delta)\ldots(\beta-\varphi)$. Idipſum quadrat in alios ceterarum conſtantium c, f &c. denominatores.

15. Praeſtantiſſimam methodum uno vel altero exemplo explanare non utile modo, ſed neceſſarium prorſus arbitramur. Eſto itaque

EXEMPLUM I.

Scala relationis $1+2$

Indices	0	1	2	3	4	5 ……	x
Series recurrens	0,	u,	u^2,	$3u^3$,	$5u^4$,	$11u^5$ …	$y_x u^x$

Aequatio differentialis finita $y_x = y_{x-1} + 2y_{x-2}$, ſeu

(A)

(A) $y_x - y_{x-1} - 2y_{x-2} = 0$

ſumatur $y_x = a z^x$, quo ſubſtituto in (A) oritur

Aequatio relationis $1 - \frac{1}{z} - \frac{2}{zz} = 0$

ſeu $zz - z - 2 = 0$,

cuius radices ſunt $\alpha = 2$, $\beta = -1$. Eſt igitur integrale completum, hoc eſt terminus generalis

$$y_x = a\alpha^x + b\beta^x = a.2^x + b(-1)^x$$

Hypotheſis I. $x = 0$; $a + b = 0$; $a = \frac{1}{3}$

Hypotheſis II. $x = 1$; $2a - b = 1$; $b = -\frac{1}{3}$.

Hiſce inventis oritur $y_x = \frac{1}{3}[2^x - (-1)^x]$; ideoque generalis ſeriei terminus erit $\frac{1}{3}[2^x - (-1)^x]u^x$. Q. E. I.

EXEMPLUM II.

Scala relationis $0 + 7 - 6$.

Indices 0 1 2 3 4 5 6 7 8 ... x.

Series recur. $0, u, u^2, 7u^3, u^4, 43u^5, -35u^6, 295u^7, -503u^8 .. y_x u^x$

Aequatio differ. finita $y_{x+3} + 0.y_{x+2} - 7y_{x+1} + 6y_x = 0$

fiat $y_x = a z^x$ & praecedens aequatio mutatur in

Aequationem relationis $z^3 \quad * \quad - 7z + 6 = 0$,

cuius tres radices ſunt $z = \alpha = 1$; $z = \beta = 2$; $z = \gamma = -3$.

Hinc habetur aequationis differentialis integrale completum

$$y_x = a\alpha^x + b\beta^x + c\gamma^x = a + b.2^x + c(-3)^x.$$

Hypotheſis I. $x = 0$; $a + b + c = 0$

II. $x = 1$; $a + 2b - 3c = 1$.

III. $x = 2$; $a + 4b + 9c = 1$.

Ex tribus hiſce aequationibus per notas algebrae regulas invenitur $a = -\frac{1}{2}$; $b = \frac{3}{5}$; $c = -\frac{1}{10}$;

ac denique $y_x = -\frac{1}{2} + \frac{3}{5}.2^x - \frac{1}{10}(-3)^x$.

Quapropter ſeriei terminus generalis $y_x u^x$ prodibit $=$

$$\left(\frac{3}{5}.2^x - \frac{1}{10}(-3)^x - \frac{1}{2}\right)u^x. \text{ Q. E. I.}$$

16. Hactenus a nobis ea tacite conditio poſita fuit, quae

quae radices omnes α, β, γ, &c. inter ſe inaequales conſtituit: At ſi radices aliquae aſſumantur aequales, valores nonnulli conſtantium arbitrariarum a, b, c &c. infiniti deprehenduntur. Revera facto $r = 3$ invenimus ſupra §. 12

$$a = \frac{\beta\gamma y_0 - (\beta+\gamma) y_1 + y_2}{(\alpha-\beta)(\alpha-\gamma)}\,;\quad b = \frac{\alpha\gamma y_0 - (\alpha+\gamma) y_1 + y_2}{(\beta-\alpha)(\beta-\gamma)}\,;$$

$$c = \frac{\alpha\beta y_0 - (\alpha+\beta) y_1 + y_2}{(\gamma-\alpha)(\gamma-\beta)}:$$

Quapropter aſſumtis aequalibus radicibus binis α, β quantitates a & b infinitae evadunt, tertia c finita perſeverante. Idem oſtenditur in aliis ipſius r hypotheſibus, ubi radices aliquae inter ſe aequales fuerint.

Huic incommodo, quod frequentiſſime occurrit, ſequens adhiberi ſolet remedium. In generali ſeriei recurrentis termino $a\alpha^x + b\beta^x + c\gamma^x +$ &c. quando $\beta = \alpha$, accipiatur $\beta = \alpha + d\alpha$, ita ut differentia inter utramque radicem ſit infinite parva; & ſumta hic ex. gr. ſerie recurrente ordinis tertii, cuius generalis terminus ſit trinomia quantitas $a\alpha^x + b\beta^x + c\gamma^x$, per ſubſtitutionem binomii $\alpha + d\alpha$ loco ipſius β terminus ipſe generalis tranſit in hunc alterum $a\alpha^x + b(\alpha+d\alpha)^x + c\gamma^x$, qui infinite parum differt a vero. Vertatur iam in ſeriem iuxta Neutonianum canonem binomium $(\alpha+d\alpha)^x$ orieturque

$$(\alpha + d\alpha)^x = \alpha^x + x\,d\alpha.\alpha^{x-1} + \frac{x(x-1)}{2}\,d\alpha^2.\alpha^{x-2} + \&c.$$

Hinc terminus generalis erit

$$\text{(D)}\quad (a+b)\alpha^x + c\gamma^x + bx\,d\alpha.\alpha^{x-1} + \frac{bx(x-1)}{2}\,d\alpha^2.\alpha^{x-2} + \&c.$$

Invenimus ſupra $b = \dfrac{\alpha\gamma y_0 - (\alpha+\gamma) y_1 + y_2}{(\beta-\alpha)(\beta-\gamma)}$; ideſt ſubſtituto $d\alpha$ loco $\beta - \alpha$ fit

$$b = \frac{\alpha\gamma y_0 - (\alpha+\gamma) y_1 + y_2}{d\alpha(\beta-\gamma)},\ \text{vel}\ \frac{\alpha\gamma y_0 - (\alpha+\gamma) y_1 + y_2}{d\alpha(\alpha-\gamma)};$$

proin-

proindeque $bd\alpha = \frac{\alpha\gamma y_0 - (\alpha+\gamma) y_1 + y_2}{\alpha - \gamma}$, quae manifeſto quantitas eſt finita. Sed ſi $bd\alpha$ finita quantitas eſt; infinite parva erit $bd\alpha^2$, multoque magis $bd\alpha^3$ &c. Quapropter generalis terminus (D) evadet =

$$(E) \qquad (a+b)\alpha^x + bxd\alpha . \alpha^{x-1} + c\gamma^x .$$

Praeterea quoniam $a = \frac{6\gamma y_0 - (6+\gamma) y_1 + y_2}{(\alpha - 6)(\alpha - \gamma)}$, per ſubſtitutionem $-d\alpha$ loco $\alpha - 6$ oritur $a = \frac{6\gamma y_0 - (6+\gamma) y_1 + y_2}{-d\alpha(\alpha - \gamma)}$, quae pariter infinita quantitas eſt. At ſi valores bini quantitatum infinitarum a & b ſimul colligantur adhibitis generalibus eorum expreſſionibus exurget aggregatum finitum. Etenim

$$a + b = \frac{6\gamma y_0 - (6+\gamma) y_1 + y_2}{(\alpha - 6)(\alpha - \gamma)} - \frac{\alpha\gamma y_0 - (\alpha+\gamma) y_1 + y_2}{(\alpha - 6)(6 - \gamma)}$$

& redactis fractionibus ad eandem denominationem invenitur

$$a + b = \frac{(6^2\gamma - 6\gamma^2 - \alpha^2\gamma + \alpha\gamma^2) y_0 + (\alpha^2 - 6^2) y_1 - (6 - \alpha) y_2}{(\alpha - 6)(\alpha - \gamma)(6 - \gamma)}$$

& diviſione facta per $\alpha - 6$ prodit

$$a + b = \frac{(\gamma^2 - \alpha\gamma - 6\gamma) y_0 + (\alpha + 6) y_1 - y_2}{(\alpha - \gamma)(6 - \gamma)}$$

expreſſio prorſus ab infinitis libera. Quare facto $6 = \alpha$ tandem aſſequimur

$$a + b = \frac{(\gamma^2 - 2\alpha\gamma) y_0 + 2\alpha y_1 - y_2}{(\alpha - \gamma)^2}$$

In termino itaque generali (E) dabuntur coefficientes quantitatum α^x & $x\alpha^{x-1}$, finitus uterque, infiniteſimis exiſtentibus coefficientibus $bd\alpha^2$, $bd\alpha^3$ &c. terminorum

$$\frac{x(x-1)}{2}\alpha^{x-2}, \quad \frac{x(x-1)(x-2)}{2.3}\alpha^{x-3} \text{ \&c.}$$

qui propterea negliguntur. Atque hinc ſi ſtatuatur $a + b = a'$;

$bd\alpha = b'$, terminus generalis induet hanc formam

$$y_x = a'\alpha^x + b'x\alpha^{x-1} + c\gamma^x$$

17. Pergo nunc ad ſeries illas recurrentes, in quibus aequatio relationis tres habet radices aequales. Aſſumo ſeriei terminum generalem

$$y_x = a\alpha^x + b\beta^x + c\gamma^x + f\delta^x,$$

in quo ſit $\alpha = \beta = \gamma$. Ut harum radicum aequalitas citra errorem e medio tollatur, infinite parvam differentiam inter ipſas conſtituo, ponendo $\beta = \alpha + d\alpha$; $\gamma = \alpha + d\beta$; ſicque generalis terminus evadit

$$y_x = a\alpha^x + b(\alpha + d\alpha)^x + c(\alpha + d\beta)^x + f\delta^x:$$

adhibito Neutoniano canone, & pro quovis binomio tribus tantum acceptis terminis, aſſequimur

$$\text{(F)}\quad y_x = (a+b+c)\alpha^x + (bd\alpha + cd\beta)x\alpha^{x-1} + (bd\alpha^2 + cd\beta^2)\frac{x(x-1)}{2}\alpha^{x-2} + f\delta^x.$$

Supra autem §. 12. invenimus

$$a = \frac{-\beta\gamma\delta y_0 + (\beta\gamma + \beta\delta + \gamma\delta)y_1 - (\beta + \gamma + \delta)y_2 + y_3}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}$$

$$b = \frac{-\alpha\gamma\delta y_0 + (\alpha\delta + \alpha\gamma + \gamma\delta)y_1 - (\alpha + \gamma + \delta)y_2 + y_3}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}$$

$$c = \frac{-\alpha\beta\delta y_0 + (\alpha\beta + \alpha\delta + \beta\delta)y_1 - (\alpha + \beta + \delta)y_2 + y_3}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}$$

$$f = \frac{-\alpha\beta\gamma y_0 + (\alpha\beta + \alpha\gamma + \beta\gamma)y_1 - (\alpha + \beta + \gamma)y_2 + y_3}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}$$

Hinc ob $\alpha = \beta = \gamma$ valores coefficientium a, b, c infiniti evadunt ordinis ſecundi; ſed valor poſtremi f finitus eſt. Quamvis autem ſinguli a, b, c infinitam magnitudinem ordinis ſecundi conſtituant, eorum tamen aggregatum finitum eſſe deprehenditur. Nam facta reductione trium valorum ad denominatorem eundem, diviſaque fractione per $(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$ invenitur tandem

$$a+b+c = \frac{(\alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta - \alpha\delta^2 - \beta\delta^2 - \gamma\delta^2 + \delta^3)y_0 - (\alpha\gamma + \alpha\beta + \beta\gamma)y_1 + (\alpha + \beta + \gamma)y_2 - y_3}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}$$

quae

quae evidenter finita quantitas eſt, & ſubſtituto α loco ipſorum β & γ, fit eadem ſumma

$$a+b+c=\frac{(-3\alpha^2\delta+3\alpha\delta^2-\delta^3)\,y_0+3\alpha^2 y_1-3\alpha\,y_2+y_3}{(\alpha-\delta)^3}$$

Praeterea in valoribus coefficientium b, c ſubſtituo $\alpha+d\alpha$ pro β, & $\alpha+d\beta$ pro γ, tum duco b in $d\alpha$, & c in $d\beta$; quo facto invenio

$$b\,d\alpha=\frac{(-\alpha^2\delta-\alpha\delta\,d\beta)y_0+[\alpha^2+2\alpha\delta+(\alpha+\delta)d\beta]\,y_1-(2\alpha+\delta+d\beta)y_2+y_3}{(d\alpha+d\beta)\,(\alpha-\delta+d\alpha)}$$

$$c\,d\beta=\frac{(\alpha^2\delta+\alpha\delta\,d\alpha)\,y_0-[\alpha^2+2\alpha\delta+(\alpha+\delta)d\alpha]\,y_1+(2\alpha+\delta+d\alpha)y_2-y_3}{(d\alpha-d\beta)\,(\alpha-\delta+d\beta)}$$

Valores hoſce ad communem denominatorem redigo, eorumque ſummam capio, quae diviſione facta per $d\alpha-d\beta$, neglectiſque terminis aliorum comparatione evaneſcentibus reperitur

$$b\,d\alpha+c\,d\beta=\frac{(2\alpha^2\delta-\alpha\delta^2)\,y_0-(2\alpha^2+2\alpha\delta-\delta^2)y_1+3\alpha\,y_2-y_3}{(\alpha-\delta)^2}$$

haecque evidenter finita quantitas eſt.

Denique multiplico $b\,d\alpha$ per $d\alpha$, & $c\,d\beta$ per $d\beta$; & duobus valoribus reſultantibus ad eandem denominationem revocatis, eorum accipitur ſumma, quae poſt diviſionem per $(d\alpha-d\beta)\,(\alpha-\delta)$, & neglectum terminorum infinite parvorum deprehenditur

$$b\,d\alpha^2+c\,d\beta^2=\frac{-\alpha^2\delta\,y_0+(\alpha^2+2\alpha\delta)\,y_1-(2\alpha+\delta)\,y_2+y_3}{\alpha-\delta}$$

nimirum aequalis quantitati finitae.

Si igitur in termino generali (F) capiatur $a+b+c=a'$; $b\,d\alpha+c\,d\beta=b'$; $b\,d\alpha^2+c\,d\beta^2=c'$, quae quidem omnes ex demonſtratis finitae quantitates ſunt, aſſequemur generalem ipſum terminum

$$y_x=a'\alpha^x+x\,b'\alpha^{x-1}+\frac{x\,(x-1)}{2}c'\alpha^{x-2}+f\delta^x.$$

In praecedenti calculo tres dumtaxat Neutoniani canonis terminos

minos in quovis binomio accepimus, eo quod termini ſubſequentes infiniteſimi ſunt, ſeu evaneſcentes; quartus enim terminus foret

$$(b\,d\,\alpha^3+c\,d\,\beta^3)\,\frac{x(x-1)(x-2)}{2.\;3}\,\alpha^{x-3},$$ ubi ob b, c infinitas quantitates ordinis ſecundi evadit coefficiens $(b\,d\,\alpha^3+c\,d\,\beta^3)$ infiniteſimus ordinis primi, proindeque terminus hic quartus negligi tuto poteſt, multoque magis quintus ceterique ſequentes.

18. Eodem omnino ratiocinandi modo generalem ſeriei recurrentis terminum $y_x = a\alpha^x + b\beta^x + c\gamma^x + f\delta^x + b\varepsilon^x +$&c. ubi quatuor ſunt radices aequales nimirum $\alpha=\beta=\gamma=\delta$ ita inveniemus expreſſum

$$y = (a+b+c+f)\alpha^x + (bd\alpha+cd\beta+fd\gamma)x\alpha^{x-1} + (bd\alpha^2+cd\beta^2+fd\gamma^2)\frac{x(x-1)\alpha^{x-2}}{2}$$
$$+(bd\alpha^3+cd\beta^3+fd\gamma^3)\frac{x(x-1)(x-2)\alpha^{x-3}}{2.\;3} + b\varepsilon^x + \text{\&c.}$$

ubi non modo coefficiens $a+b+c+f$ oſtenditur abſolute finitus, ſed finiti quoque demonſtrantur coefficientes reliqui trinomiales

$bd\alpha + cd\beta + fd\gamma$, $bd\alpha^2 + cd\beta^2 + fd\gamma^2$; $bd\alpha^3 + cd\beta^3 + fd\gamma^3$, quamvis ſpeciem mentiantur quantitatum infinite parvarum.

In hac autem hypotheſi quatuor radicum aequalium non pergitur ultra quartum Neutoniani canonis terminum, eo quod quintus, & a fortiori ſextus ac ſequentes omnes termini ob infinitam parvitatem evaneſcant. Quocirca aſſumi poterit termini α^x coefficiens $a+b+c+f=a'$; termini $x\alpha^{x-1}$ coefficiens $bd\alpha+cd\beta+fd\gamma=b'$; termini $\frac{x(x-1)\alpha^{x-2}}{2}$ coefficiens $bd\alpha^2+cd\beta^2+fd\gamma^2=c'$; denique termini $\frac{x(x-1)(x-2)\alpha^{x-3}}{2.\;3}$ coefficiens $bd\alpha^3+cd\beta^3+fd\gamma^3=f'$; & generalis ſeriei terminus y_x vertetur in hunc

$y_x =$

$$y_x = a'\alpha^x + xb'\alpha^{x-1} + \frac{x(x-1)}{2}c'\alpha^{x-2} + \frac{x(x-1)(x-2)}{2.3}f'\alpha^{x-3} + b\varepsilon^x + \&c.$$

Ex hactenus dictis quid fieri oporteat in exemplis omnibus, ubi plures quam quatuor aequales aequationis radices occurrunt, nemo non intelligit; lex eadem enim ſemper perſeverat, ac totum ſupputationis negotium moderatur. Non inutile erit peculiarem hanc radicum aequalium hypotheſim unico ſaltem exemplo illuſtrare.

EXEMPLUM.

Eſto recurrens ſeries

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & \ldots & x \\ 1 + 8u & + 27u^2 & + 64u^3 & + 125u^4 & + 216u^5 & \ldots & \ldots & y_x u^x. \end{matrix}$$

quae evolvitur ex rationali fractione $\frac{1+4u+u^2}{(1-u)^4}$. Manifeſtum eſt fore ſcalam relationis $4 - 6 + 4 - 1$; proindeque terminum generalem

$$y_x = 4y_{x-1} - 6y_{x-2} + 4y_{x-3} - y_{x-4};$$

unde oritur aequatio finita differentialis

$$y_x - 4y_{x-1} + 6y_{x-2} - 4y_{x-3} + y_{x-4} = 0.$$

Fiat $y_x = az^x$, & praedicta aequatio mutabitur in aequationem relationis

$$1 - \frac{4}{z} + \frac{6}{z^2} - \frac{4}{z^3} + \frac{1}{z^4} = 0,$$

$$\text{ſeu} \quad z^4 - 4z^3 + 6z^2 - 4z + 1 = 0,$$

quae evidenter nihil aliud eſt niſi $(z-1)^4 = 0$. Igitur aequatio relationis praedita eſt radicibus quatuor aequalibus $z = \alpha = \beta = \gamma = \delta = 1$. Quare in formula termini generalis mox inventa

$$y_x = a'\alpha^x + b'x\alpha^{x-1} + c'\frac{x(x-1)}{2}\alpha^{x-2} + f'\frac{x(x-1)(x-2)}{2.3}\alpha^{x-3}$$

ſubſtituendo unitatem pro α nanciſcemur

$$y_x = a' + b'x + c'\frac{x(x-1)}{2} + f'\frac{x(x-1)(x-2)}{2.3}$$

$= \qquad = a'$

$$= a' + \left(b' - \frac{c'}{2} + \frac{f'}{3}\right)x + \left(\frac{c' - f'}{2}\right)x^2 + \frac{f'}{6}x^3;$$

captoque rurſus $b' - \frac{c'}{2} + \frac{f'}{3} = b''$; $\frac{c' - f'}{2} = c''$; $\frac{f'}{6} = f''$

oritur tandem terminus generalis

$$y_x = a' + b''x + c''x^2 + f''x^3$$

Quatuor porro arbitrariae conſtantes a', b'', c'', f'' determinantur accipiendo pro x ſucceſſive 0, 1, 2, 3, ex quo habentur valores termino generali y_x reſpondentes 1, 8, 27, 64, & conſequenter aequationes quatuor ſequentes

$$a' = 1$$
$$a' + b'' + c'' + f'' = 8$$
$$a' + 2b'' + 4c'' + 8f'' = 27$$
$$a' + 3b'' + 9c'' + 27f'' = 64,$$

ex quibus deducitur $a' = 1$; $b'' = 3$; $c'' = 3$; $f'' = 1$. Quo facto habetur $y_x = 1 + 3x + 3x^2 + x^3 = (1+x)^3$;

Adeoque $y_x u^x = (1+x)^3 u^x$. Q. E. I.

Ceterum in hoc exemplo idem obtinetur alia methodo ab Auctore noſtro tradita. Et ſane ſi ſeriei terminus generalis vocetur Y; terminique quatuor ipſum praecedentes $'Y$, $''Y$, $'''Y$, $''''Y$, ex enunciata conditione habemus $Y = 4\,'Y - 6\,''Y + 4\,'''Y - ''''Y$, hoc eſt $''''Y - 4\,'''Y + 6\,''Y - 4\,'Y + Y = 0$, ſeu terminis ſingulis ad quarta quaque loca ſubſequentia translatis prodibit aequatio $Y - 4Y' + 6Y'' - 4Y''' + Y'''' = 0 = \Delta^4 Y$ (§. 23. Cap. I.). Quare eo res redit ut finita differentialis aequatio $\Delta^4 Y = 0$ integretur in hypotheſi quod Δx conſtans ſit & unitati aequalis. Iam vero ex praecedentibus conſtat

$$\Sigma\Delta^4 Y = \Delta^3 Y = a\Delta x$$
$$\Sigma\Delta^3 Y = \Delta^2 Y = \Sigma a\Delta x = ax + b\Delta x$$
$$\Sigma\Delta^2 Y = \Delta Y = \Sigma ax + bx + c\Delta x = \frac{ax^2}{2} - \frac{ax}{2} + bx + c\Delta x$$
$$\Sigma\Delta Y = Y = \Sigma\frac{ax^2}{2} - \Sigma\frac{ax}{2} + \Sigma bx + cx + f$$

=

$$= \frac{a}{6}x^3 + \frac{b-a}{2}x^2 + \left(\frac{a}{3} - \frac{b}{2} + c\right)x + f$$

Igitur ſi capiatur $\frac{a}{6} = f''$; $\frac{b-a}{2} = c''$; $\frac{a}{3} - \frac{b}{2} + c = b''$; $f = a'$, habebitur, prorſus ut ante, generalis ſeriei terminus

$$Y = a' + b''x + c''x^2 + f''x^3,$$

Qui de integratione formularum differentialium finitarum plura cupit, adeat inſignes Geometras La Grange, La Place, Monge, Condorcet, &c., qui in Actis Academicis Taurinenſibus, Berolinenſibus, Pariſinis hac de re copioſe exquiſiteque tractarunt.

ADNOT. poſt CAP. III. PART. I.

Ut haec nova Auctoris noſtri theoria de quantitatum infiniteſimarum, atque infinitarum natura mutuaque relatione ad germanum legitimumque ſenſum revocetur, iiſque omnibus difficultatibus & controverſiis praecludatur aditus, quas eadem ab Auctore vix dum prolata ubique excitavit & peperit, operae pretium erit ipſam interpretari, atque explicare perinde ac ſi a Newtoniana celeberrima *rationum primarum & ultimarum*, vel *limitum* doctrina vix, aut ne vix quidem differret. Sequentia itaque conſtituenda ſunt principia ac dictata.

I. Si quantitas quaedam A continuo creſcendo vel decreſcendo ad aliam L magnitudinem datam, quam nunquam aequare poteſt, magis magiſque accedit, ita ut ab ea differre poſſit magnitudine utcunque parva, atque adeo minori quam quaelibet data; vocabitur quantitas ipſa L *limes* quantitatis prioris A.

Sic I. Curvae cuiuſcunque tangens *limes* eſt omnium ſecantium.

II. Fractio $\frac{1}{3}$ *limes* eſt fractionis huius decimalis

$0,\ 333333333$ &c.

III. Unitas *limes* eſt ſummae huius progreſſionis geometricae

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+ \text{\&c. in infin.}$$

IV. Quantitas $a + \frac{1}{3}a$ *limes* eſt ſummae huius progreſſionis

$$a+\frac{a}{4}+\frac{a}{16}+\frac{a}{64}+\frac{a}{256}+ \text{\&c. in inf.}$$

V. Radix quadrata numeri binarii *limes* eſt rationis illius, quam habet quadrati diagonalis ad latus.

VI. Denique ſumma geometricae progreſſionis decreſcentis

$$a+b+\frac{b^2}{a}+\frac{b^3}{a^2}+\frac{b^4}{a^3}+ \text{\&c. in inf.}$$

limitem habet $= \frac{a^2}{a-b}$; nimirum (quod & de praecedentibus exemplis tenendum eſt) progreſſionis ſumma ita ſemper magis magiſque accedit ad valorem $\frac{a^2}{a-b}$, ut quamvis eum nunquam adaequet, differre tamen ab eodem poſſit quantitate quantumvis parva, & quavis data minori: hoc autem vel ex eo liquet, quod ſi progreſſio non in infinitum continuetur, terminuſque eius poſtremus ſit p, ſummam recipit omnino accuratam & exactam $\frac{a^2-bp}{a-b}$, quae ſemper minor eſt quam $\frac{a^2}{a-b}$, earumque differentia fieri poteſt utcunque parva, quia terminus ipſe p per ulteriorem progreſſionis continuationem evadit utcunque parvus. Atque hinc quemadmodum p ſemper propius accedit ad nihilum ſeu o, quod tamen nunquam adaequat, atque adeo nihilum ipſum ſeu o *limes* eſt termini p, ita pariter fractio $\frac{a}{a-b}$ *limes* eſt fractionis $\frac{a^2-bp}{a-b}$, quae cum

cum priori non congruit, niſi fiat $p = 0$ hoc eſt niſi pro p *limes* ipſius ſufficiatur.

2. Sit y functio quantitatis variabilis x per $Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. repraeſentata, ita ut poſito $x = 0$, fiat $y = 0$, & ex aequatione $y = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. per x diviſa oriatur $\frac{y}{x} = A + Bx + Cx^2 + Dx^3 +$ &c. Patet itaque rationem $\frac{y}{x}$ variabilem eſſe quia exponens huius rationis eſt quantitas variabilis, nimirum functio ipſius variabilis x, & quidem talis quae iugiter decreſcit quamdiu ipſa variabilis x minuitur. Poſito iam $x = 0$ functio ipſa y ſimul evadit $= 0$, & rationis $\frac{y}{x}$ exponens $= A$, ſeu ipſa ratio $\frac{y}{x} = A$. Paradoxum videtur quod hic aſſeritur eſſe $\frac{y}{x} = \frac{0}{0} = A$ quia cyphrae non ſunt quantitates, neque una altera maior vel minor eſſe dicive poteſt. Verum ratio $\frac{y}{x} = A$ revera non eſt ratio cyphrarum, ſed *limes*, ad quem ratio $\frac{y}{x}$ perpetuo appropinquat, variabili x & cum ea ſimul variabili y continuo decreſcente; *limes* quem nondum attingit ratio $\frac{y}{x}$ quamdiu x & y ſunt quantitates verae quantumvis parvae. Ratio $\frac{y}{x} = A$ ea eſt quacum variabiles x & y evaneſcunt, ſeu quantitates eſſe deſinunt; atque hic aequationis *limes* certus eſt ac definitus. Poteſt hic *limes* rationis $\frac{y}{x} = A$ vocari *ultima ratio* variabilium y & x, ſi ſpectetur ut ea, quacum y & x quantitates eſſe deſinunt ſeu evaneſcunt. Si variabilis

riabilis $x = 0$ iterum creſcere concipiatur, functio y cum ea ſimul creſcere incipit, atque ratio $\frac{y}{x} = A$ iterum ea erit, quacum creſcere incipiunt; atque hoc reſpectu *limes* huius rationis $\frac{y}{x}$, qui eſt A, vocari poteſt prima ratio variabilium y & x, quacum quantitates eſſe incipiunt. Hoc modo ad verum germanumque ſenſum revocabitur Auctoris noſtri doctrina contendentis infinite parva revera eſſe nihila, interque duo infinite parva intercedere poſſe rationem quamcunque. Ratio infinite parvorum revera non eſt ratio nihilorum, ſed *ratio ultima*, quacum duae variabiles naſcuntur, ſeu quantitates eſſe incipiunt. Hoc eſt aliis verbis ratio infinite parvorum nihil aliud eſt, quam *limes* ille, ad quem quantitatum variabilium ratio perpetuo accedit, quem nunquam tranſgredi, neque etiam attingere poteſt, ſed ad quem propius accedit pro data quavis differentia. Verbo dicam: *ultimae* magnitudinum evaneſcentium *rationes*, & *primae* naſcentium non ſunt rationes, quas magnitudines ipſae unquam inter ſe habent, ſed *limites* omnium variabilium rationum. In paradoxa expreſſione $\frac{0}{0}$ poſſumus *limitem* latentem detegere hac ratiocinatione: eſt $\frac{0}{0} = \frac{a-a}{b-b}$, eſt autem $a : b :: x : \frac{bx}{a}$, igitur $a + x : b + \frac{bx}{a} :: a : b$, & $a + x - a : b + \frac{bx}{a} - b :: a : b$. Quocirca erit $\frac{a}{b} = \frac{a-a+x}{b-b+\frac{bx}{a}}$; Hinc nulleſcente x rationis conſtantis $\frac{a}{b}$ *limes* aequabitur ipſi $\frac{a-a}{b-b} = \frac{0}{0}$.

3.

3. Omnis variabilis quantitas non tantum continuo decreſcere, & tandem penitus evaneſcere poteſt, ſed etiam continuo augeri poteſt, idque in infinitum. Eſto

$$y = A + Bx + Cx^2 \ldots + Hx^{n-1} + Kx^n$$

in hac aequatione apparet, aucto in infinitum x etiam y in infinitum excreſcere. Ex ea autem deducitur

$$\frac{y}{x^n} = \frac{A}{x^n} + \frac{B}{x^{n-1}} + \frac{C}{x^{n-2}} + \ldots + \frac{H}{x} + K;$$

ex quo patet rationem $\frac{y}{x^n}$ eiuſque exponentem variabilem eſſe, ita quidem ut variabili x, & cum ea ſimul ipſa y perpetuo creſcente, exponens illius continuo decreſcat, terminis nimirum omnibus praeter ultimum conſtantem decreſcentibus. Poſito autem $x = \infty$, qua in hypotheſi fit etiam $y = \infty$, exponens rationis $\frac{y}{x^n}$ evadit aequalis conſtanti K, reliquis, terminis ob $x = \infty$, nulleſcentibus. Verum, ficuti antea pro quantitatibus infinite parvis monuimus, ratio iſtaec $\frac{y}{x^n} = K$ proprie loquendo non eſt ratio infinite magnorum y & x^n ad ſe invicem, quia fieri nequit, ut duo quanta inaſſegnabilia inter ſe comparentur; ſed eſt *limes* rationis, ad quem variabiles y & x^n in infinitum creſcentes continuo appropinquant, quemque nunquam attingunt quandiu ſunt determinatae & aſſignabiles quantitates; atque idcirco $\frac{y}{x^n} = K$ eſt *ultima ratio*, quacum quantitates y, & x^n aſſignabiles eſſe definunt. Hinc recte intelligitur pervulgatum illud Calculi Infiniteſimalis dictatum, quantitates ſcilicet, & quantitatum rationes, quarum data ſit differentia, ſi ita perpetuo excreſcant, ut datam quamvis magnitudinem ſuperent *ultimo* fieri aequales. Sint enim A, B quantitates ſeu rationes ſine fine creſcentes, quarum differentia

data

data ſemperque conſtans ſit D. Capio magnitudinem aliam datam & conſtantem a, ſtatuoque analogiam $A : D :: a : d$. Perſpicuum eſt creſcente A decreſcere d, & quidem proportionaliter, ita ut ſi creſcat A ultra quamlibet datam quantitatem, minuatur d infra quamlibet datam. Igitur ob $A \pm D : D :: a \pm d : d$, quum quantitates duae a & $a \pm d$, decreſcente d infra quamlibet datam quantitatem, *ultimo*, ſeu in *limite* ubi nimirum nulleſcit, fiant aequales; etiam quantitates proportionales A, & $A \pm D$, ſeu A & B datam habentes differentiam D *ultimo* ad aequalitatem pervenient. Hinc quemadmodum per quantitates & rationes perpetuo decreſcentes & quibuſcunque datis minores determinantur *proportiones ultimae*, quae valent in *limite* continue decreſcentium; ita per quantitates & rationes ſine fine creſcentes, & quibuslibet datis maiores definiuntur *proportiones ultimae*, quae praeſto ſunt in *limite* perpetuo creſcentium. Veruntamen inter duos hoſce *limites* extremos illud intercedit diſcrimen, quod decreſcentium *limitem* aſſequi, & detegere licet, ubi quantitates prorſus evaneſcunt nullaeque evadunt; ſed *limitem* creſcentium nunquam proprie aſſequimur, quia tametſi quantitates terminum quemcunque datum creſcendo praetergredi poſſunt, nunquam tamen fieri poſſunt *abſolute* infinitae.

4. Illud itaque fixum ratumque ſemper habeatur in Calculo Differentiali nunquam aequales aſſumi eas quantitates, aut rationes, quarum datur aliqua utcunque parva differentia, ſed illas duntaxat, quarum differentia ultra quemlibet terminum aſſignabilem decreſcens, tandem evaneſcere, nullaque prorſus fieri demonſtratur, ita ut quantitatum, & rationum aequalitas obtineatur tantum in *limite extremo*, ad quem eaedem ultra quamcunque menſuram accedunt, & quamvis ſaepiſſime contingat, ut quantitates ipſae, quarum inveſtigatur proportio in *limite* prorſus evaneſcant, nihilo tamen minus *limitis* proportio, in quem unice inquiritur, ſemper ſubſiſtit, & immutata perſeverat.

5.

5. Nomine itaque *Differentiarum*, *Fluxionum*, *Elementorum*, *Incrementorum*, *Decrementorum*, *Quantitatum Infiniteſimarum*, quomodocunque appellentur, nihil aliud ſignificatum volumus, niſi differentias continuo decreſcentes, & quibuſlibet datis minores a veteribus ipſis Geometris uſurpatas, ex quibus aequalitates & proportiones *ultimas*, quae locum habent in *limite*, tuta conſecutione deducimus. Brevitatis autem gratia utimur compendio, & *limitis* aequalitatem tribuimus quantitatibus, rationibuſque differentiam habentibus perpetuo decreſcentem & inaſſignabilem, non quod revera aequales ſtatuamus quantitates & rationes, donec differentiam aliquam utcunque minimam ſervant, ſed quod alias notum perſpectumque ſit, differentiam in *limite* ipſo nimirum *ultimam* nullam fieri, adeoque quantitates ac rationes in *limite* eſſe accuratiſſime aequales. Quapropter magnam Calculo Differentiali inſerret iniuriam ſi quis contenderet, differentias, quae adhuc aliquid ſunt, ab ipſo negligi, & pro nullis haberi. Nihil procul dubio in infiniteſimorum Analyſi, nihil omnino contemnitur: nam cum aequales pronunciamus quantitates & rationes infiniteſimas ſeu inaſſignabili differentia praeditas, non intelligimus eas aequales eſſe, donec differentiam habent aliquam utcunque parvam, ſed in *limite*, ubi differentia prorſus nulla eſt, earum abſolutiſſimam aequalitatem concludimus. Quocirca infiniteſimas differentias negligere, & pro nullis omnino haberi, nihil aliud ſibi vult, quam *limitis* abſolutiſſimas aequalitates ac proportiones inquirere ac definire. Hinc autem perſpicue patet *Differentialem Calculum* nihil aliud eſſe, & definiri, quam analyticam methodum inveniendi *limitem* rationis, quae intercedit inter differentiam finitam duarum quantitatum, & differentiam finitam aliarum binarum, quae habent ad priores duas analogiam & relationem lege cognitam ac definitam.

6. Huic adnotationi finem imponat graviſſimum magni Fluxionum inventoris **Newtoni** teſtimonium hac de re ita ap-

aptiſſime mirificeque differentis (*a*): „ Malui demonſtrationes ad ultimas quantitatum evaneſcentium ſummas & rationes, primaſque naſcentium, id eſt ad limites ſummarum & rationum deducere; & propterea limitum illorum demonſtrationes qua potui brevitate praemittere. Proinde in ſequentibus, ſi quando quantitates tanquam ex particulis conſtantes conſideravero, vel ſi pro rectis uſurpavero lineolas curvas; nolim indiviſibilia, ſed evaneſcentia diviſibilia, non ſummas & rationes partium determinatarum, ſed ſummarum & rationum limites ſemper intelligi „.

„ Obiectio eſt, quod quantitatum evaneſcentium nulla ſit ultima proportio; quippe quae antequam evanuerunt non eſt ultima; ubi evanuerunt nulla eſt. Sed & eodem argumento aeque contendi poſſet, nullam eſſe corporis ad certum locum, ubi motus finiatur, pervenientis velocitatem ultimam: hanc enim, antequam corpus attingit locum, non eſſe ultimam; ubi attingit nullam eſſe. Et reſponſio facilis eſt: Per velocitatem ultimam intelligi eam, qua corpus movetur, neque antequam attingit locum ultimum & motus ceſſat, neque poſtea, ſed tunc cum attingit; id eſt illam ipſam velocitatem, quacum corpus attingit locum ultimum, & quacum motus ceſſat. Et ſimiliter per ultimam rationem quantitatum evaneſcentium, intelligendam eſſe rationem quantitatum, non antequam evaneſcunt, non poſtea, ſed quacum evaneſcunt. Pariter & ratio prima naſcentium eſt ratio quacum naſcuntur. Et ſumma prima & ultima eſt quacum eſſe (vel augeri aut minui) incipiunt & ceſſant. Extat limes, quem velocitas in fine motus attingere poteſt, non autem tranſgredi. Haec eſt velocitas ultima. Et par eſt ratio limitis quantitatum & proportionum omnium incipientium & ceſſantium. Cumque hic limes ſit certus & definitus, problema eſt vere geometricum eundem determinare. Geometrica vero omnia in aliis geometricis determi-

(*a*) *Phil. Nat. Princ. Math. Lib. I. Sect. I. in fine.*

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terminandis ac demonſtrandis legitime uſurpantur „.

„ Contendi etiam poteſt, quod ſi dentur ultimae quantitatum evaneſcentium rationes, dabuntur & ultimae magnitudines: & ſic quantitas omnis conſtabit ex indiviſibilibus, contra quam *Euclides* de incommenſurabilibus, in libro decimo Elementorum demonſtravit. Verum haec obiectio falſae innititur hypotheſi. Ultimae rationes illae, quibuſcum quantitates evaneſcunt, revera non ſunt rationes quantitatum ultimarum, ſed limites, ad quos quantitatum ſine limite decreſcentium rationes ſemper appropinquant; & quas propius aſſequi poſſunt quam pro data quavis differentia, nunquam vero tranſgredi, neque prius attingere quam quantitates diminuuntur in infinitum. Res clarius intelligitur in infinite magnis. Si quantitates duae, quarum data eſt differentia, augeantur in infinitum, dabitur harum ultima ratio, nimirum ratio aequalitatis, nec tamen ideo dabuntur quantitates ultimae ſeu maximae quarum iſta eſt ratio. In ſequentibus igitur, ſi quando facili rerum conceptui conſulens dixero quantitates quam minimas, vel evaneſcentes, vel ultimas; cave intelligas quantitates magnitudine determinatas, ſed cogita ſemper diminuendas ſine limite „.

7. Offendit Alembertum Neutoniana loquendi formula ab aliis etiam recepta nimirum *ratio quacum quantitates evaneſcunt*; *ratio quacum naſcuntur*: locutionem hanc quaſi abſonam & abſurdam his verbis notat & carpit (*a*): „ Quelques mathematiciens ont défini la quantité infiniment petite celle qui, s'évanouit, conſidérée non pas avant qu'elle s'évanouiſſe, non pas après qu'elle eſt évanouie, mais dans le moment même où elle s'évanouit. Je voudrois bien ſavoir quelle idée nette & preciſe on peut eſpérer de faire naitre dans l'eſprit par une ſemblable définition. Une quantité eſt quelque choſe ou rien. Si elle eſt quelque choſe, elle n'eſt pas évanouie; ſi

Fff ff elle

(*a*) *Mel. de Lit. d'Hiſt. & de Phil. Tom. V. Eclairc. ſur les Elém. de Phil.* §. *XIV*.

elle eſt rien, elle eſt évanouie tout-a-fait. C' eſt une chimere que la ſuppoſition d' un état moyeu entre ces deux-la „.

At equidem non video quid iſtud ſit, quod ſummo Geometrae Alemberto tantopere faſtidium creat: huiuſmodi enim loquendi formulae, licet fortaſſis ad ſeveriſſimam veritatis normam minus exactae, ſunt tamen a Geometris, atque adeo Philoſophis communi uſu ac ſermone receptae, ut longiores verborum ambitus vitentur. Quid quod alibi Alembertus ipſe id genus formulas adeo non improbet, ut potius brevitatis & compendii cauſſa commendet? „ Toutes les parties des mathématiques „ inquit ille (*a*) „ ſont ſouvent uſage d' expreſſion de cette eſpece, qui dans le ſens metaphyſique qu' elles preſentent, paroiſſent d' abord peu exactes; mais qui ne doivent être regardées que comme des manieres abrégées de s' exprimer, que les Mathématiciens ont inventées pour énoncer une verité, dont le developpement & l' énoncé exact auroit demandé beaucoup de mots „.

(*a*) *Loc. cit.* §. *XI.*

ADNOT. ad §. 214. *PART. I.*

QUoniam Auctoris noſtri demonſtratio mera nititur inductione, adeoque ſumma illa ſeveritate caret, quam ſemper Geometrae, ubi res fieri poteſt, requirunt; demonſtrationem ipſam ab Alemberto mutuabimur, qui *Tom. IV. Opuſ. Mathem. pag.* 5. eam hiſce verbis adornat.

„ Si une quantité A contient tant de variables x, y, z, &c. qu' on voudra, & qu' on la différentie en faiſant varier ſucceſſivement x, y, z, &c. en négligeant les différences ſecondes, troiſiémes &c. on aura le même reſultat dans quelqu' ordre qu' on différen-

diſſérencie, c' eſt-à-dire, que par exemple, $\frac{d^n A}{dx\, dy\, dz\, dt\ \&c.}$ $= \frac{d^n A}{dz\, dy\, dt\, dx\ \&c.}$. M. Euler a démontré cette propoſition dans ſon *Analyſe des infiniment petits*, mais par une eſpece d' induction. Pour en donner une démonſtration générale & rigoureuſe nous conſidérerons, I. que $\frac{ddA}{dx\, dy} = \frac{ddA}{dy\, dx}$, comme le ſavent les Géomètres. II. Nous allons démontrer que ſi en général les quantités $\frac{d^n A}{dx\, dy\, dz}$, & $\frac{d^n A}{dz\, dx\, dy}$, ſont égales, la même égalité ſubſiſtera en faiſant varier une nouvelle variable t; ce qu' il eſt aiſé de voir en conſidérant I. que la combinaiſon $dx\, dy\, dz$, donne (*hyp.*) le même réſultat que la combinaiſon $dz\, dx\, dy$, la combinaiſon $dt\, dx\, dy\, dz$ donne évidemment le même réſultat que $dt\, dz\, dx\, dy$; II. que $dt\, dx\, dy\, dz$ donne le même réſultat que $dx\, dt\, dy\, dz$, puiſque $dt\, dx$ donne le même que $dx\, dt$; III. que $dx\, dt\, dy\, dz$ donne le même réſultat que $dx\, dy\, dt\, dz$, & par la même raiſon puiſque $dt\, dy$ donne le même que $dy\, dt$, &c. Donc, &c. Donc puiſque le théorême a lieu lorſque $n = 2$, il aura lieu lorſque $n = 3$, & enſuite lorſque $n = 4$, &c. & ainſi de ſuite „.

Hanc demonſtrationem applicat Alembertus communi quantitatum algebraicarum multiplicationi, qua de re elementorum Scriptores monet, eorumque circumſpectionem excitat his verbis:

„ Cette démonſtration pourroit ſervir à prouver d' une maniere très-ſimple une propoſition que la plupart des Auteurs élémentaires négligent de prouver, ſavoir, qu' en quelqu' ordre qu' on multiplie tant de quantités a, b, c, d, e *&c.* qu' on voudra, les unes par les autres, le réſultat eſt toujours le même. On le démontre bien pour les produits ab & ba de deux quantités, mais on néglige ſouvent de le prouver pour les

produits d'un plus grand nombre de quantités, quoique la choſe ne ſoit pas évidente par elle-même. C'eſt un avis qu'on donne ici aux Auteurs d'Elémens, afin qu'ils y faſſent attention à l'avenir „.

ADNOT. ad CAP. VIII. PART. I.

Celeber. De la Grange in italica epiſtola ad Fagnanum anno 1754 typis edita novam proponit ſeriem pro differentialibus, & integralibus cuiuslibet gradus, eamque adamuſſim reſpondentem & analogam Neutonianae ſeriei pro poteſtatibus & radicibus. Rem novitate atque elegantia commendabilem hic breviter explanabimus: Sint itaque ſeries binae Neutoniana una, Grangiana altera:

$$\text{I. } (a+b)^m = a^m b^0 + m a^{m-1} b^1 + \frac{m(m-1)}{2} a^{m-2} b^2 + \frac{m(m-1)(m-2)}{2.\ 3} a^{m-3} b^3 + \&c.$$

$$\text{II. } (xy)^m = x^m y^0 + m x^{m-1} y^1 + \frac{m(m-1)}{2} x^{m-2} y^2 + \frac{m(m-1)(m-2)}{2.\ 3} x^{m-3} y^3 + \&c.$$

Iam vero I. quemadmodum prima ſeries dat evolutionem poteſtatis cuiuſcunque, ad quam elevatur ſumma duarum, adeoque quotcunque datarum magnitudinum, aſſumto exponente m pro gradu propoſitae poteſtatis; ita propemodum ſeries ſecunda praebet differentiale cuiuſcunque gradus ex producto duarum, adeoque quotcunque, variabilium, ſumto pariter exponente m pro gradu vel ordine differentialis propoſiti.

II. Quemadmodum prima ſeries ſuppeditat terminos radicis cuiuſcunque, quae a ſumma duarum, vel quotcunque quantitatum, extrahitur, modo fiat exponens m aequalis numero fracto radicis gradum indicanti; ita ſeries ſecunda continet integrale cuiuslibet gradus ex producto duarum, vel quotcunque finitarum, aut infinitesimarum magnitudinum, dum-

dummodo accipiatur exponens m aequalis numero integro, ſed negativo gradum dati integralis indicanti.

III. Denique ſicuti in prima ſerie exponens o indicat quantitatem, quam afficit, elevatam ad poteſtatem nullam, adeoque aequalem unitati; ita in ſecunda exponens idem o non aliud ſignificat, quam in quantitate ab eodem affecta nullum eſſe differentiationi, nec integrationi locum, adeoque quantitatem ipſam talem accipiendam, qualis nulla habita exponentis ratione reperitur.

Propterea ſi Neutoniana ſerie feliciter utimur per univerſam Analyſim in elevationibus ad poteſtates, & in radicum cuiuſcunque gradus extractionibus; ſeriem alteram ſucceſſu aeque proſpero in differentiationibus & integrationibus cuiuſcunque ordinis uſurpabimus. Proponatur igitur quantitas xy differentianda: in hoc caſu, quoniam differentiale quaeſitum primi ordinis eſt, erit $m = 1$, proindeque generalis ſeries, ſecunda induet hanc formam $x^1y^0 + x^0y^1$, quae ad communem notandi modum redacta iuxta id, quod ante indicavimus, convertitur in hanc $ydx + xdy$.

Si ſecundum, aut tertium, aut quartum differentiale quaeratur erit $m = 2$, vel $m = 3$, vel $m = 4$, & quaeſita differentialia, factis pro m debitis ſubſtitutionibus erunt ſequentia:

$d^2.xy = x^2y^0 + 2x^1y^1 + x^0y^2 = y\,ddx + 2dx\,dy + x\,ddy$;

$d^3.xy = x^3y^0 + 3x^2y^1 + 3x^1y^2 + x^0y^3 = yd^3x + 3ddx\,dy + 3dx\,ddy + xd^3y$;

$d^4.xy = x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + x^0y^4 =$
$y\,d^4x + 4d^3x\,dy + 6ddx\,ddy + 4dx\,d^3y + x\,d^4y$:

Sicque porro differentiale cuiuſcunque ordinis ſumma facilitate ſemper obtinebitur, ſpectato tanquam variabili etiam differentiali primo dx, vel ſecus deletis terminis, qui differentialia ipſius dx continent.

Sed praeterea ſeries eadem II. ad integralium inventionem aeque accommodatur, ac Neutoniana ſeries ad radicum extractionem. Proponatur ex. gr. ad habendam quadraturam indefinitam curvae cuiuslibet integratio elementi ydx areae curvi-

lineae:

lineae: accipiatur in canone generali $dx = x$; eritque per id, quod ante adnotavimus, $m = -1$, quibus valoribus in canone ſecundo ſubſtitutis aſſequemur ſeriem peculiarem

$$dx^{-1}y^{0} - dx^{-2}y^{1} + dx^{-3}y^{2} - dx^{-4}y^{3} + dx^{-5}y^{4} \ \&c.$$

Iam vero dx^{-1} denotat integrale ipſius dx; dx^{-2} integrale integralis ipſius dx (ideſt integrale quantitatis x) quod voco integrale ſecundum eiuſdem dx, deſignoque per ſymbolum ${}^{2}\!\int dx$: Pariter dx^{-3} exprimit integrale tertium ipſius dx, nimirum ${}^{3}\!\int dx$; dx^{-4} integrale quartum, ſeu ${}^{4}\!\int dx$; &c. Sed

$$\int dx = x;$$

$${}^{2}\!\int dx = \int x = \int \frac{x\,dx}{dx} = \frac{x^{2}}{2\,dx};$$

$${}^{3}\!\int dx = \int \frac{x^{2}}{2\,dx} = \int \frac{x^{2}\,dx}{2\,dx^{2}} = \frac{x^{3}}{2.3\,dx^{2}};$$

$${}^{4}\!\int dx = \int \frac{x^{3}}{2.3\,dx^{2}} = \int \frac{x^{3}\,dx}{2.3\,dx^{3}} = \frac{x^{4}}{2.3.4\,dx^{3}};$$

& in univerſum

$${}^{m}\!\int dx = \frac{x^{m}}{2.3.4 \ldots m\,dx^{m-1}},$$ accepto ſemper dx tanquam conſtanti. Igitur hiſce valoribus in praecedenti ſerie ſubrogatis, poſitiſque de more dy, ddy, $d^{3}y$, &c. loco y^{1}, y^{2}, y^{3}, &c. tandem apparebit

$$\int y dx = xy - \frac{x^{2}\,dy}{2\,dx} + \frac{x^{3}\,ddy}{2.3\,dx^{2}} - \frac{x^{4}\,d^{3}y}{2.3.4\,dx^{3}} + \frac{x^{5}\,d^{4}y}{2.3.4.5\,dx^{4}} - \&c.$$

haecque eſt decantata ſeries Bernoulliana (a) Geometris dudum notiſſima.

Ceterum hic canon ſecundus non ſolum ad differentialia primi gradus integranda extenditur, verum etiam unica operatione integralia differentialium cuiuslibet ulterioris gradus confeſtim ſuppeditat. Quaeratur ex. gr. integrale ſecundum producti $dy\,dx$; facto igitur $m = -2$, $x = dx$, & $y = dy$ conſequemur

(a) Io. Bern. *Oper. Tom. I.* N. *XXI.*

$$^2\!\int dy\, dx = dx^{-2} dy^0 - 2dx^{-3} dy^1 + 3 dx^{-4} dy^2 - 4 dx^{-5} dy^3 + \&c.$$

$$= \frac{x^2 dy}{2 dx} - \frac{2x^3 ddy}{2.3\, dx^2} + \frac{3x^4 d^3y}{2.3.4 dx^3} - \frac{4x^5 d^4y}{2.3.4.5 dx^4} + \&c.$$

Quum autem ob dx conſtantem ſit $^2\!\int dy\, dx = \int y dx$, & ante inventum ſit integrale

$$\int y\, dx = xy - \frac{x^2 dy}{2 dx} + \frac{x^3 ddy}{2.3 dx^2} - \frac{x^4 d^3y}{2.3.4 dx^3} + \&c.;$$

aequabuntur idcirco inter ſe binae ſeries ſpecie tenus immaniter diſcrepantes, hoc eſt

$$xy - \frac{x^2 dy}{2 dx} + \frac{x^3 ddy}{2.3\, dx^2} - \&c. = \frac{x^2 dy}{2 dx} - \frac{2x^3 ddy}{2.3 dx^2} + \frac{3x^4 d^3y}{2.3.4 dx^3} - \&c.$$

quarum paullo ocultior aequalitas vel ex eo colligitur, quod ſi utraque bis differentietur, termini omnes utrinque deſtruuntur unico utrobique manente $dydx$.

ADNOT. ad §. 324. *PART, I.*

Data differentiali aequatione tres variabiles involvente

(A) $\qquad P dx + Q dy + R dz$

deductaque inde *aequatione conditionis*

(B) $$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = 0,$$

Auctor noſter canonem hunc generatim conſtituit, quod nimirum ubi aequatio finita (B) nec identica reperitur, nec eam praebet variabilium x, y, z relationem, quae aequationi differentiali (A) ſatisfaciat, *nulla exhiberi poſſit aequatio finita*, quae ipſi ſatisfaciat. At vero canonem hunc non ea gaudere extenſione, quam Eulerus ei tribuit, ſed circumſcriptione aliqua coercendum eſſe primus acutiſſime animadvertit Celeberr. De la Place in egregia Diſſertatione *Sur les ſolutions particulieres*

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lieres des Equations différentielles, & ſur les inégalités ſéculaires des Planetes, In Actis Reg. Scient. Pariſ. Acad. an. 1772. Si enim differentialis aequatio fuerit

$$dx = \{1 + \sqrt[3]{(x-z-y)}\,[z + a\sqrt[4]{(x-z-y)} + b\sqrt[3]{(x-z-y)}]\}\,dy$$
$$+ [1 + y\sqrt[3]{(x-z-y)}]\,dz,$$

quae cum aequatione (A) collata praebet

$$P = 1,\ Q = 1 + \sqrt[3]{(x-z-y)}\,[z + a\sqrt[4]{(x-z-y)} - b\sqrt[3]{(x-z-y)}],$$
$$R = 1 + y\sqrt[3]{(x-z-y)};$$

atque hi valores ſubſtituantur in aequatione conditionis (B), eruitur poſt moleſtam prolixamque ſupputationem aequatio linearis

$y + z - x + \left(\frac{3a}{4b}\right)^{12} = 0$, quae nec identica eſt, nec differentiali aequationi ſatisfaciens. At nihilominus datur inter variabiles x, y, z relatio, quae hoc ipſum praeſtat; ſiquidem $x = y + z$ propoſitae differentiali aequationi evidentiſſime ſatisfacit. Ex quo luculenter patet, Eulerianum Canonem brevioribus terminis contineri quam ab Auctore ſancitur.

ADNOT. ad §. 11. *PART. II.*

Auctor noſter in hoc loco teſtatur ſe in Introductione oſtendiſſe, *ſeriem utramque*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$$

$$\& \ \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \frac{1}{6.64} + \&c.$$

logarithmum hyperbolicum binarii exhibere. At in citata *Introductio-*

ductione ad Analyſim Infinitorum deſideratur demonſtratio ab EULERO indicata, nec eam ibi invenire omni adhibita diligentia mihi contigit. Quapropter eandem hic ſupplere non abs re erit. Conſtat ex logarithmorum theoria, eſſe

$$l(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.$$

adeoque poſito $x = 1$ erit

$$l\,2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$$

Eſt autem aſſumto $x = -\frac{1}{2}$

$$l(1+x) = l\,\frac{1}{2} = -\frac{1}{1.2^1} - \frac{1}{2.2^2} - \frac{1}{3.2^3} - \frac{1}{4.2^4} - \&c.$$

Igitur ob $l\,2 = -l\,\frac{1}{2}$, orietur

$$l\,2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$$

$$= \frac{1}{1.2^1} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1}{4.2^4} + \frac{1}{5.2^5} + \&c.$$

Q. E. D.

ADNOT. poſt §§. 45. & ſeqq. PART. II.

QUamvis inſigne ac mirificum Taylorianum Theorema innumeris prope uſibus per matheſim univerſam accommodetur, & elegantiſſimas pariat, vixque aliunde ſperandas Problematum etiam difficillimorum ſolutiones, ſubliniorumque Theorematum demonſtrationes; quandoque tamen uſuvenit, ut niſi caute & circumſpecte adhibeatur in errorem ac fallaciam inducat; fallitur ſcilicet qui in univerſum affirmat

 functio-

functionem quamlibet binomiae quantitatis $x + a$, hoc eſt $\varphi(x+a)$ per Taylorianum Theorema ſemper aſſumi poſſe aequalem indefinitae ſeriei

$$\varphi(x) + \frac{a d.\varphi(x)}{dx} + \frac{a^2 dd.\varphi(x)}{2dx^2} + \frac{a^3 d^3.\varphi(x)}{2.3\, dx^3} + \frac{a^4 d^4.\varphi(x)}{2.3.4\, dx^4} + \&c.$$

Hoc enim vel in ſimpliciſſima hypotheſi, qua ſtatuatur $\varphi(x+a) = [\sin(x+a)]^{\frac{1}{3}}$, falſum prorſus & abſurdum deprehenditur, quum quantitas $[\sin(x+a)]^{\frac{1}{3}}$, quae ſemper manifeſto finita eſt in Taylorianam ſeriem converſa, vel in ipſo ſecundo termino valorem adipiſcatur infinitum ubi $x = 0$, fit quippe feries =

$$\sin x^{\frac{1}{3}} + \frac{ad. \sin x^{\frac{1}{3}}}{dx} + \&c. = \sin x^{\frac{1}{3}} + \frac{a \cos x}{3 \sin x^{\frac{2}{3}}} + \&c. = \infty$$

in hypotheſi $x = 0$. Vel hoc uno edoctus exemplo magnam ſemper adhibebit prudens Geometra conſiderationem & ſagacitatem in applicatione atque uſu Tayloriani Theorematis, quod incaute uſurpatum nonnullis etiam magni nominis Geometris non ſemel impoſuit.

Accidit etiam quandoque, ut ex Tayloriano Theoremate, licet citra errorem uſurpato, ſolutio propoſiti problematis habeatur non ita generica lateque patens, qualem eſſe oporteret, vel qualem methodus alia tutior ac melior ſuppeditaret. En luculentum huiuſce rei exemplum. Proponatur invenienda talis functio $\varphi(x)$ quantitatis variabilis x, ut habeatur $\varphi(x+a) = \varphi(x)$, exiſtente a quantitate data. Palam eſt a formula $\varphi(x)$ exhiberi ordinatam eius curvae, in qua, acceptis in axe ſegmentis $= a$, ordinatae per hanc quantitatem aequidiſtantes aequales ſint: id accidit in Cycloide ſi a aequetur circuli genitoris circumferentiae. Patet igitur quantitatem $\varphi(x)$ infinitis praeditam eſſe valoribus arbitrariis problemati ſatisfacientibus, dummodo $\varphi(x)$ praedictae curvae ordinatam exhibeat: Vertamus nunc $\varphi(x + a)$ in ſeriem per canonem Taylorianum, eritque

$\varphi(x)$

$$\phi(x)=\phi(x+a)=\phi(x)+\frac{a d.\phi(x)}{dx}+\frac{a^2 dd.\phi(x)}{2dx^2}+\frac{a^3 d^3.\phi(x)}{2.3\,dx^3}+\&c.$$

atque hinc habebitur aequatio

$$\frac{a d.\phi(x)}{dx}+\frac{a^2 dd.\phi(x)}{2dx^2}+\frac{a^3 d^3.\phi(x)}{2.3dx^3}+\&c.=0.$$

Sit e logarithmorum hyperbolicorum baſis, & accipiatur $\phi(x)=Ae^{bx}$, huiuſque valoris fiat in praecedenti aequatione ſubſtitutio. Quocirca diviſione facta per Ae^{bx} exurgit aequatio $ab+\frac{a^2b^2}{2}+\frac{a^3b^3}{2.3}+\&c.$ Erit igitur $e^{ab}-1=0$; adeoque facto regreſſu a numeris ad logarithmos orietur $ab=l1$. Sed per notiſſimum Euleri noſtri inventum eſt $l1=\pm m\pi\sqrt{-1}$, exiſtente m numero quovis integro una cum o. Igitur $ab=\pm m\pi\sqrt{-1}$, & $b=\pm\frac{m\pi\sqrt{-1}}{a}$, & conſequenter

$$\phi(x)=Ae^{\pm\frac{m\pi x}{a}\sqrt{-1}}=A\left(\cos\frac{m\pi x}{a}\pm\sin\frac{m\pi x}{a}.\sqrt{-1}\right),$$

ſeu poſito $A\sqrt{-1}=B$ aſſequimur tandem

$$\phi(x)=A\cos\frac{m\pi x}{a}\pm B\sin\frac{m\pi x}{a}.$$

Iam vero haec functionis $\phi(x)$ forma, ſeu valor non videtur eam univerſalitatem habere, quam oportet: capiatur enim circuli arcus v, & diametri abſciſſa $1-\cos v$, erigaturque ordinata x ad eam curvam, in qua habeatur $x=v-\sin v$; ſitque adeo curvae abſciſſa $1-\cos v$ aequalis functioni ipſius ordinatae x, ſeu $\phi(x)$. Erit itaque

$$x=v-\sin v=Cv^3+Dv^5+Ev^7+\&c.,$$

& per ſerierum regreſſum invenietur

$$v=Fx^{\frac{1}{3}}+\&c.;\ \text{proindeque}\ \phi(x)=1-\cos v=1-\cos(Fx^{\frac{1}{3}}+\&c.).$$

Si iam valor ſupra inventus $\phi(x)=A\cos\frac{m\pi x}{a}\pm B\sin\frac{m\pi x}{a}$

debita donaretur univerſalitate, is in ſe contineret valorem alterum $1 - \cos(Fx^{\frac{1}{3}} + \&c.)$, ſeu quod idem eſt quantitas $1 - \cos(Fx^{\frac{1}{3}} + \&c.)$ converti poſſet in ſeriem huius formae $A' \sin \pi x + B' \sin 2\pi x + C' \sin 3\pi x + \&c. \ldots + H' \cos \pi x + I' \cos 2\pi x + \&c.$ adeoque (per nota Trigonometriae analyticae Theoremata ;) in ſeriem

$$A'' + B''\pi x + C''\pi^2 x^2 + D''\pi^3 x^3 + E''\pi^4 x^4 + \&c.$$

in qua nulla adeſt ipſius variabilis x poteſtas niſi integra. Atqui hoc repugnat evolutioni formulae $1 - \cos(Fx^{\frac{1}{3}} + \&c.)$, quae praebet ut notum eſt $A''' x^{\frac{2}{3}} + B''' x^{\frac{4}{3}} + \&c.$, ubi ipſius x poteſtates fractae evitari nullo modo poſſunt. Patet igitur functionis $\varphi(x)$ valorem $A \cos \frac{m\pi x}{a} \pm B \sin \frac{m\pi x}{a}$ exprimendis omnibus quaeſtionis propoſitae caſibus nequaquam aptari, minuſque genericum eſſe quam oporteat

Si quis regerat quantitatem $A''' x^{\frac{2}{3}} + B''' x^{\frac{4}{3}} + \&c.$ revocari poſſe ad formam $\alpha + \beta \cos 2x + \gamma \cos 4x + \&c.$, propterea quod habeatur

$$x = \sin x + a \sin x^3 + b \sin x^5 + \&c. = \sin x (1 + a \sin x^2 + b \sin x^4 + \&c.)$$
$$= \sin x (a' + b' \cos 2x + c' \cos 4x + \&c.);$$ ideoque
$$x^2 = \frac{1 - \cos 2x}{2} (a'' + b'' \cos 2x + c'' \cos 4x + \&c.);$$ ac denique
$x^{\frac{2}{3}} = a''' + b''' \cos 2x + c''' \cos 4x + \&c.$: ſi quis inquam hoc regerat, is refelletur ea maxime ratione, quod aequatione

$$x^{\frac{2}{3}} = a''' + b''' \cos 2x + c''' \cos 4x + \&c.$$

differentiata diviſaque per dx oritur aequatio altera

$$\frac{2}{3x^{\frac{1}{3}}} = -2b''' \sin 2x - 4c''' \sin 4x - \&c.$$

quae

quae in hypotheſi $x=0$ abſurdiſſimam parit aequalitatem $\infty=0$. Praeterea ſi propoſita ſtaret oppoſitio, etiam $\sin x$ converti poſſet in $f+g\cos 2x+h\cos 4x+$ &c. liceret quippe ſtatuere

$$\sin x=\sqrt{\sin x^2}=\sqrt{\left(\frac{1-\cos 2x}{2}\right)}=f+g\cos 2x+h\cos 4x+\text{\&c.},$$

quae tamen aequatio duplici vitio laborat: I. Quod $\frac{d.\sin x}{dx}=1$ quando $x=0$, & contra $\frac{d(f+g\cos 2x+h\cos 4x+\text{\&c.})}{dx}=0$ quando $x=0$; II. Quod $y=\sin x$ curvam deſignat oppoſitis ramis inſtructam ſupra & infra axem ſitis; contra vero

$$y=f+g\cos 2x+h\cos 4x+\text{\&c.}$$

curvam definit, cuius rami abſciſſis x & $-x$ reſpondentes ad eandem axis partem iacent.

ADNOTATIO ad §. 69. *PART. II.*

INter plures, quas Auctor noſter inſtituit Tayloriani Theorematis applicationes ad functionum cuiuſcunque variabilis evolutionem, eos tantummodo caſus expendit, in quibus variabilis ipſa certum ſuſcipit incrementum vel decrementum; nullatenus vero evolutionem conſiderat ubi variabilis immutata perſeverat, quamvis Tayloriana formula in his quoque caſibus mirum praeſtet uſum & inſigne adiumentum. Id breviter explanare ad maiorem formulae illius admirabilis amplificationem non abs re erit.

Itaque ſit z functio binomii $x+\omega$, & y ſimilis functio ipſius x, hoc eſt $z=\varphi(x+\omega)$, & $y=\varphi(x)$; erit

$$\text{(A)}\qquad z=y+\frac{\omega\,dy}{dx}+\frac{\omega^2\,ddy}{2\,dx^2}+\frac{\omega^3\,d^3y}{2.3\,dx^3}+\frac{\omega^4\,d^4y}{2.3.4\,dx^4}+\text{\&c.},$$

quae ſane aequatio ſemper vera erit, quaecunque aſſumatur quan-

quantitas pro ω, & quaecunque ſit variabilis x. Igitur ſubſiſtet aequatio (A) tum etiam, cum in omnibus eius terminis per x & ω datis poſitum fuerit $x = 0$, & deinde $\omega = x$. Quocirca ſi valores, in quos migrant termini

$$y,\ \frac{dy}{dx},\ \frac{ddy}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4}\ \&c.$$

per ſubſtitutionem $x = 0$, dicantur K, A, B, C, D, &c., nanciſcimur formulam

$$\text{(B)}\qquad \varphi(x) = K + Ax + \frac{Bx^2}{2} + \frac{Cx^3}{2.3} + \frac{Dx^4}{2.3.4} + \&c.$$

EXEMPLUM I.

Evolvenda proponatur quantitas exponentialis a^x

Aſſumto itaque $x = 0$ invenitur

$K = 1$, $A = la$, $B = (la)^2$, $C = (la)^3$, $D = (la)^4$, &c. Quamobrem opportunis factis ſubſtitutionibus in aequatione (B) erit $a^x = 1 + x\,la + \frac{x^2 (la)^2}{2} + \frac{x^3 (la)^3}{2.3} + \frac{x^4 (la)^4}{2.3.4} + \&c.$

EXEMPLUM II.

Quaeratur ſin x per ſeriem poteſtatum arcus x.

Erit itaque $\sin x = K + Ax + \frac{Bx^2}{2} + \frac{Cx^3}{2.3} + \frac{Dx^4}{2.3.4} + \&c.$

eſt autem $K = 0$, $A = 1$, $B = 0$, $C = -1$, $D = 0$, $E = 1$, &c.

Igitur $\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$

ADNOT. poſt §. 151. PART. II.

IN ſeriebus huiuſmodi reciprocis poteſtatum numerorum naturalium, quae generaliter repraeſentantur per

$$\text{(A)}\qquad \frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \frac{1}{6^m} + \&c.\ \text{in inf.}$$

illud

illud habetur certo ſingulare, quod *ſumma terminorum omnium in locis imparibus ſe habet ad ſummam terminorum in locis paribus uti* $2^m - 1$ *ad* 1: Nam diviſa ſerie (A) per 2^m oritur ſeries

$$(B) \quad \frac{1}{2^m} + \frac{1}{4^m} + \frac{1}{6^m} + \frac{1}{8^m} + \frac{1}{10^m} + \frac{1}{12^m} + \&c.$$

quae terminos continet ſeriei (A) in locis paribus exiſtentes. Eſt autem $\frac{(A)}{2^m} = (B)$; & conſequenter $(A) : (B) :: 2^m : 1$. Igitur $(A) - (B) : (B) :: 2^m - 1 : 1$, hoc eſt:

$$\frac{1}{1^m} + \frac{1}{3^m} + \frac{1}{5^m} + \&c. : \frac{1}{2^m} + \frac{1}{4^m} + \frac{1}{6^m} + \&c. :: 2^m - 1 : 1,$$

nimirum ſumma terminorum in locis imparibus eandem habet rationem ad ſummam terminorum in locis paribus, quam habet $2^m - 1$ ad 1. Atque ita poſito exponente $m = 1$ iuvenitur ſeries reciproca imparium aequalis ſeriei reciprocae parium nimirum

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c. = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \&c.;$$

Poſito $m = 2$ invenitur

$$\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \&c. : \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} \&c. :: 3 : 1;$$

Sumto $m = 3$ detegitur

$$\frac{1}{1} + \frac{1}{27} + \frac{1}{125} + \frac{1}{343} + \frac{1}{729} + \&c. :$$

$$\frac{1}{8} + \frac{1}{64} + \frac{1}{216} + \frac{1}{512} + \frac{1}{1000} + \&c. :: 7 : 1;$$

& in univerſum in ſeriebus, quae fiunt ex poteſtatibus inverſis numerorum naturalium, ſumma terminorum in locis imparibus tantundem excedit ſummam terminorum in locis paribus quantum binarii poteſtas homologa unitate mulctata excedit unitatem.

Verum hic inopinato inſigne ſe nobis offert paradoxum, quod

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quod nimirum quotieſcunque poteſtatis exponens non eſt maior unitate ſumma terminorum in locis imparibus per demonſtratum Theorema minor ſit oportet quam ſumma terminorum in locis paribus; quum tamen contra termini locorum imparium cum terminis parium ſingillatim collati ſemper inveniantur manifeſto maiores: quippe in praecedenti analogia $\frac{1}{1^m}+\frac{1}{3^m}+\frac{1}{5^m}+\frac{1}{7^m}+\frac{1}{9^m}+$ &c.:

$\frac{1}{2^m}+\frac{1}{4^m}+\frac{1}{6^m}+\frac{1}{8^m}+\frac{1}{10^m}+$ &c. :: $2^m-1:1$, ratio $2^m-1:1$ eſt ſemper *minoris inaequalitatis*, ſeu ratio minoris ad maius, quoties exponens *m* minor eſt unitate; & tamen ratio altera $\frac{1}{1^m}+\frac{1}{3^m}+\frac{1}{5^m}+$ &c.: $\frac{1}{2^m}+\frac{1}{4^m}+\frac{1}{6^m}+$ &c. eſt *maioris inaequalitatis*, ſeu ratio maioris ad minus, quia terminus quilibet in antecedente manifeſto excedit terminum quemlibet ſibi reſpondentem in conſequente.

Paradoxum hoc non fugit acutiſſimum Iacobum Bernoullium, qui in tractatu de Seriebus Infinitis §. XXIV. haec habet: „Mirabile vero eſt quod in ſerie $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+$ &c., cuius ſumma infinita eſt, ſeu maior ſerie $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. ob denominatores minores, termini locorum imparium ad terminos parium iuxta regulam inveniuntur habere rationem $\sqrt{2}-1$ ad 1, minoris ſcilicet ad maius, cum tamen illi cum his ſingillatim collati iiſdem manifeſto ſint maiores, cuius ἐναντιοφανείας rationem, etſi ex infiniti natura finito intellectui comprehendi non poſſe videatur, nos tamen ſatis perſpectam habemus. Idem vero de ſimilibus ſeriebus aliis, quae infinitam ſummam habent intelligendum „. Sed quum veram legitimamque paradoxi explicationem, quam

ſe

ſe tenere aiebat, Bernoullius nunquam vulgaverit, eamque in poſthuma eius operum collectione incaſſum quaeſierim; non inutile arbitror eam hic aperire. Itaque quando ſeriei

(A) $\frac{1}{1^m}+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\frac{1}{5^m}+\frac{1}{6^m}+\frac{1}{7^m}+\frac{1}{8^m}+$ &c.

dividuntur termini omnes per 2^m, ut inde oriatur ſeries altera

(B) $\frac{1}{2^m}+\frac{1}{4^m}+\frac{1}{6^m}+\frac{1}{8^m}+\frac{1}{10^m}+\frac{1}{12^m}+\frac{1}{14^m}+\frac{1}{16^m}+$&c.

certum eſt, huiuſce terminos in locis paribus ſeriei (A) reperiri, quemadmodum tertiae ſeriei

(C) $\frac{1}{1^m}+\frac{1}{3^m}+\frac{1}{5^m}+\frac{1}{7^m}+\frac{1}{9^m}+\frac{1}{11^m}+$ &c.

termini in locis eiuſdem imparibus inveniuntur; attamen numerus terminorum ſeriei (B) bis excedit numerum terminorum ſeriei (C), quia ſinguli ſeriei (B) termini oriuntur a ſingulis ipſius (A) diviſis per 2^m; contra vero ſeriei (C) termini ſinguli reſpondent alternis tantum terminis prioris (A), eorumque adeo numerus dimidium duntaxat eſt numeri terminorum in (A) & proinde dimidium etiam numeri terminorum in (B). Fit hinc ut in mutua ſerierum (C) & (B) comparatione non ſinguli termini cum ſingulis conferri debeant, ſed poſt inſtitutam termini primi cum termino primo comparationem terminus quilibet ſeriei (C) conferri ſemper debeat cum duobus ſimul ſeriei (B), quo facto patebit in hypotheſi $m<1$ terminum quemlibet ſeriei (C) minorem eſſe aggregato duorum ſibi reſpondentium in (B). Etenim ſit $\frac{1}{a^m}$ terminus unus quicunque ſeriei (C), & bini eidem reſpondentes in (B) erunt $\frac{1}{(2a-2)^m}$, & $\frac{1}{(2a)^m}$, & ratio illius ad aggregatum iſtorum erit $\frac{1}{a^m}:\frac{1}{(2a-2)^m}+\frac{1}{(2a)^m}$. Si porro hu-

 ius

ius rationis antecedens & conſequens per $(2a)^m$ multiplicatur; oritur analogia

$$\frac{1}{a^m} : \frac{1}{(2a)^m} + \frac{1}{(2a-2)^m} :: 2^m : 1 + \left(\frac{a}{a-1}\right)^m$$, in qua ratio altera $2^m : 1 + \left(\frac{a}{a-1}\right)^m$ eſt evidenter ratio minoris ad maius, quia ob $m < 1$ fit $2^m < 2$, & ob $\frac{a}{a-1} > 1$ fit $\left(\frac{a}{a-1}\right)^m > 1$, ac proinde $1 + \left(\frac{a}{a-1}\right)^m > 2 > 2^m$. Igitur ſinguli ſeriei (C) termini (excepto primo) cum binis ſimul ſeriei (B) collati iiſdem ſemper minores deprehenduntur, quotieſcunque exponens m deficit ab unitate, & conſequenter ſeries ipſa (C) minor eſt altera (B), prouti mutua illarum proportio ante inventa $2^m - 1 : 1$ manifeſtat.

Sed in ſerie harmonica

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

illud eſt animadverſione dignum, quod ſiquidem termini locorum imparium, excepto primo, ſemper auferantur a binis reſpondentibus ſimul iunctis locorum parium, nova inde ſeries conſurgat, quae ſumma donatur finita: quod quidem in aliis ſeriebus, ubi idem occurrit paradoxum, haudquaquam obtinet, quum in iſtis ſumma eius ſeriei, quae praedicto modo progignitur, infinita deprehendatur. Sic in ſerie harmonica ſi ab aggregato duorum terminorum $\frac{1}{4} + \frac{1}{6}$ auferatur $\frac{1}{3}$, oritur

$\frac{1}{4} + \frac{1}{6} - \frac{1}{3} = \frac{1}{2.6}$; pariter $\frac{1}{8} + \frac{1}{10} - \frac{1}{5} = \frac{1}{4.10}$;

rurſus $\frac{1}{12} + \frac{1}{14} - \frac{1}{7} = \frac{1}{6.14}$;

atque

atque ita deinceps: hincque nova colligitur ſeries

$$\frac{1}{2.6}+\frac{1}{4.10}+\frac{1}{6.14}+\frac{1}{8.18}+\frac{1}{10.22}+\text{\&c. in inſ.}$$

cuius ſummam finitam eſſe ex eo conſtat, quod ſemiſſis eſt ſeriei iſtius

$$\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\frac{1}{8.9}+\frac{1}{10.11}+\text{\&c.}$$

longe minoris hac alia

$$\frac{1}{2.2}+\frac{1}{4.4}+\frac{1}{6.6}+\frac{1}{8.8}+\frac{1}{10.10}+\text{\&c.},$$

quae ut alias exploratum eſt ſumma donatur finita, eaque admodum parva. Inde autem intelligitur, cur in ſerie harmonica ſumma terminorum in locis imparibus non ſit minor ſumma terminorum in locis paribus, quamvis terminus quilibet locorum imparium (excepto primo) minor eſt duobus reſpondentibus ſimul ſumtis locorum parium; id ſcilicet ideo contingit, quod prior ſumma deficit ab altera per quantitatem duntaxat finitam, quum tamen utraque ſumma ſit infinita.

Nec diſſimili modo ratio redditur paradoxi alterius quod nimirum in ſerie numerorum naturalium ad poteſtatem quamcunque elevatorum

$$1^m+2^m+3^m+4^m+5^m+6^m+\text{\&c. in inſ.}$$

ſumma terminorum in locis paribus habent ad ſummam terminorum omnium ſeriei rationem quam habet 2^m ad unitatem ſcilicet duplam in ſerie numerorum naturalium, quadruplam in ſerie quadratorum, octuplam in ſerie cuborum, &c. quamvis termini locorum parium, utpote partes totius ſeriei, quantitatem illius multiplicem nullo modo conſtituere poſſe videantur. Fallax quippe & anceps partis, & totius notio hiſce exemplis accommodata eodem iure adſurdum immane ſuaderet, ſeries quadratorum, cuborum, aliarumque altiorum poteſtatum ex numeris naturalibus minores eſſe oportere ſerie ipſa numerorum naturalium, in qua continentur, tanquam partes

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in toto, quod quam abſonum, & praepoſterum ſit nemo non intelligit.

ADNOTATIO ad §. 159. PART. II.

Ex elegantiſſima Auctoris noſtri aequatione

$$l1+l2+l3\ldots+lx=\frac{1}{2}l2\pi+(x+\tfrac{1}{2})lx-x+\frac{A}{1.2x}-\frac{B}{3.4x^3}+\frac{C}{5.6x^5}-\frac{D}{7.8x^7}+\&c.$$

nonnulla inferuntur theoremata, quae cum utilitate, uſu, praeſtantia ſe potiſſimum commendent, hic peculiariter proponi, demonſtrarique merentur.

Theorema I.

Sumto x *infinite magno*, & e *pro logarithmorum hyperbolicorum baſi habetur* $1.2.3.4\ldots\ldots x=\frac{x^{x+\frac{1}{2}}\sqrt{2\pi}}{e^x}$.

Euleriana praedicta aequatio ſumto $x=\infty$, & exiſtente $-x=-le^x$ tranſit in hanc

(A) $l1+l2+l3\ldots\ldots+lx=\frac{1}{2}l2\pi+(x+\frac{1}{2})lx-le^x$.

Haec autem facto tranſitu a logarithmis ad numeros convertitur in $1.2.3.4\ldots\ldots x=\frac{x^{x+\frac{1}{2}}\sqrt{2\pi}}{e^x}$. Q. E. D.

Theorema II.

Sumtis x & p *infinitae magnis aio fore*

$$x(x-1)(x-2)(x-3)\ldots(x-p+2)(x-p+1)=\frac{x^{x+\frac{1}{2}}e^{-p}}{(x-p)^{x-p+\frac{1}{2}}}.$$

In aequatione (A) ſubſtituo $x-p$ loco ipſius x, & nanciſcor

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$$l1+l2+l3\ldots+l(x-p)=(x-p+\tfrac{1}{2})\,l(x-p)-le^{x-p}+\tfrac{1}{2}\,l\,2\pi;$$

Aequationem hanc aufero a priore (A) & invenio

$$lx+l(x-1)+l(x-2)\ldots+l(x-p+2)+l(x-p+1)=(x+\tfrac{1}{2})\,lx.$$
$$-(x-p+\tfrac{1}{2})\,l(x-p)+l\,e^{-p}$$

Quamobrem progrediendo a logarithmis ad numeros apparebit.

$$x(x-1)(x-2)\ldots(x-p+2)(x-p+1)=\frac{x^{x+\frac{1}{2}}e^{-p}}{(x-p)^{x-p+\frac{1}{2}}}.\ \text{Q. E. D.}$$

Theorema III.

Captis x *&* p *numeris ingentibus (ſi infiniti caperentur aequalitas foret abſoluta) coefficiens termini* (p + 1).*ceſimi in binomio ad poteſtatem* x *elevato aequatur proxime expreſſioni*

$$\frac{x^{x+\frac{1}{2}}}{p^{p+\frac{1}{2}}(x-p)^{x-p+\frac{1}{2}}\sqrt{2\pi}}$$

Ex vulgari Algebra conſtat elevato binomio ad poteſtatem x coefficientem termini $(p+1)$.ceſimi nihil aliud eſſe quam $\frac{x(x-1)(x-2)(x-3)\ldots(x-p+1)}{1.2.3.4\ldots\ldots p}$. Sed in hypotheſi x & p infinitorum, vel etiam ingentium huius fractionis numerator per Theor. II eſt $=\frac{x^{x+\frac{1}{2}}e^{-p}}{(x-p)^{x-p+\frac{1}{2}}}$; & denominator per Theor. I. eſt $=\frac{p^{p+\frac{1}{2}}\sqrt{2\pi}}{e^{p}}$: Igitur fractio praedicta; hoc eſt coefficiens termini $(p+1)$.ceſimi in binomio ad poteſtatem x elato proxime aequatur quantitati

$$\frac{x^{x+\frac{1}{2}}}{p^{p+\frac{1}{2}}(x-p)^{x-p+\frac{1}{2}}\sqrt{2\pi}}.\ \text{Q. E. D.}$$

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COROL. I. Si fiat $p = nx$ orietur

$$\frac{x^{x+\frac{1}{2}}}{p^{p+\frac{1}{2}}(x-p)^{x-p+\frac{1}{2}}\sqrt{2\pi}} = \frac{x^{x+\frac{1}{2}}}{(nx)^{nx+\frac{1}{2}}(x-nx)^{x-nx+\frac{1}{2}}\sqrt{2\pi}}$$

$$= \frac{x^{x+\frac{1}{2}}}{n^{nx+\frac{1}{2}}x^{x+\frac{1}{2}}n^{\frac{1}{2}}(1-n)^{x-nx+\frac{1}{2}}\sqrt{2\pi}} =$$

$$= \frac{1}{n^{nx+\frac{1}{2}}n^{\frac{1}{2}}(1-n)^{x-nx+\frac{1}{2}}\sqrt{2\pi}} = \frac{1}{n^{nx}(1-n)^{x-nx+\frac{1}{2}}\sqrt{nx}.\sqrt{2\pi}}$$

$$= \frac{1}{\left(\frac{n}{1-n}\right)^{nx}(1-n)^{x+\frac{1}{2}}\sqrt{nx}.\sqrt{2\pi}}$$

COR. II. Hinc eruitur valor coefficientis maximi aut medii in binomio ad immenſam poteſtatem x elevato; nam poſito $p = \frac{1}{2}x$, ſeu $n = \frac{1}{2}$, hocque ſubſtituto in praecedenti expreſſione, mutatur illa in hanc alteram $\frac{2^x\sqrt{2}}{\sqrt{\pi x}}$, quae valorem maximi coefficientis repraeſentat.

COR. III. In binomio ad infinitam poteſtatem x elevato ratio coefficientis maximi ad ſummam coefficientium omnium aequatur quantitati $\sqrt{\frac{2}{\pi x}}$; eſt enim ut liquet ſumma omnium cofficientium $= 2^x$; proindeque praedicta ratio

$$= \frac{2^x\sqrt{2}}{\sqrt{\pi x}} : 2^x = \sqrt{\frac{2}{\pi x}}.$$

ADNO-

ADNOTATIO ad §. 185. *PART. II.*

I. Suſpecta non paucis apparebit Auctoris noſtri demonſtratio, quae tota eo nititur principio, quod proponit his verbis: *Quodſi ergo* x *fuerit numerus infinitus; quoniam is eſt neque par neque impar, haec conſideratio ceſſare debet, ac proinde in ſumma termini ambigui ſunt reiiciendi: unde ſequitur, huiuſmodi ſerierum in infinitum continuatarum ſummam exprimi per ſolam quantitatem conſtantem adiiciendam.* Nemo non videt quam labile & ruinoſum ſit principium huiuſmodi ex arcana obſcuriſſimaque numeri infiniti indole depromtum. Itaque operae pretium ducimus Eulerianas formulas nova, eaque apodictica demonſtratione munire (*a*). Sint iam ſequentes formulae

$$\text{I.}\quad 1-2+3-4+\&c.=\frac{1}{4}$$

$$\text{II.}\quad 1^2-2^2+3^2-4^2+\&c.=0$$

$$\text{III.}\quad 1^3-2^3+3^3-4^3+\&c.=-\frac{2}{16}$$

$$\text{IV.}\quad 1^4-2^4+3^4-4^4+\&c.=0$$

$$\text{V.}\quad 1^5-2^5+3^5-4^5+\&c.=\frac{16}{64}$$

$$\text{VI.}\quad 1^6-2^6+3^6-4^6+\&c.=0$$

$$\text{VII.}\quad 1^7-2^7+3^7-4^7+\&c.=-\frac{272}{256}$$

$$\text{VIII.}\quad 1^8-2^8+3^8-4^8+\&c.=0$$

$$\text{IX.}\quad 1^9-2^9+3^9-4^9+\&c.=\frac{7936}{1024}$$

&c. 2.

(*a*) *Vide* Cl. GREG. FONTANAE *Diſſert.* De Seriebus *in T. II. Part. I.* Memorie di Matematica e Fiſica della Società Italiana. 1784.

2. Praemitto Lemmata bina ſequentia.

Lemma I.

Poſito x arcu quolibet circuli radio 1 deſcripti habetur aequatio $\cos x + \cos 2x + \cos 3x + \cos 4x +$ *&c in infin.* $= -\frac{1}{2}$.

Fiat $S = \cos x + \cos 2x + \cos 3x + \cos 4x +$ &c. & multiplicetur per $\cos x$, ita ut prodeat
$S\cos x = \cos x^2 + \cos x \cos 2x + \cos x \cos 3x + \cos x \cos 4x +$ &c.
Liquet ex Trigonometria productum ex coſinibus duorum angulorum aequari dimidio coſinui ſummae ipſorum angulorum una cum dimidio coſinu eorum differentiae. Igitur reſoluto producto quolibet praedictae aequationis in terminos binos oritur.
$S\cos x = \frac{1}{2}(\cos 2x + 1) + \frac{1}{2}(\cos 3x + \cos x) + \frac{1}{2}(\cos 4x + \cos 2x) +$ &c.
$= \frac{1}{2} + \frac{1}{2}\cos x + \cos 2x + \cos 3x + \cos 4x +$ &c. $= \frac{1}{2} - \frac{1}{2}\cos x + S$.
Quamobrem erit

$$S(1 - \cos x) = \frac{1}{2}\cos x - \frac{1}{2}, \text{ ideſt } S = -\frac{1}{2}. \text{ Q. E. D.}$$

Lemma II.

Infinita ſeries $S = \sin x + \sin 2x + \sin 3x + \sin 4x +$ *&c. in infin. aequalis eſt huic expreſſioni* $\frac{\sin x}{2(1 - \cos x)}$

Ducatur propoſita ſeries in $\cos x$ & prodibit
$S\cos x = \sin x \cos x + \sin 2x \cos x + \sin 3x \cos x + \sin 4x \cos x +$ &c.
Iam vero datis binis angulis φ, θ ex theoria functionum angularium conſtat fore $\sin\varphi \cos\theta = \frac{1}{2}\sin(\varphi + \theta) + \frac{1}{2}\sin(\varphi - \theta)$.
Quapropter facta termini cuiuslibet reſolutione in duos invenietur
$S\cos x = \frac{1}{2}(\sin 2x + 0) + \frac{1}{2}(\sin 3x + \frac{1}{2}\sin x)$
$+ \frac{1}{2}(\sin 4x + \sin 2x) + \frac{1}{2}(\sin 5x + \sin 3x) +$ &c.
$= \frac{1}{2}$

$= \frac{1}{2}\sin x + \sin 2x + \sin 3x + \sin 4x + \&c. = S - \frac{1}{2}\sin x.$

Hinc per tranſpoſitionem erit $S - S\cos x = \frac{1}{2}\sin x$, ac poſtremo

$$S = \frac{\sin x}{2(1-\cos x)}. \text{ Q. E. D.}$$

3. His praemiſſis demonſtratio Formularum Eulerianarum ſponte conſequitur.

I. Differentietur ſeries Lemmatis ſecundi, dividaturque per $-dx$, ex quo orietur

(M) $$-\cos x - 2\cos 2x - 3\cos 3x - 4\cos 4x - \&c. \text{ in inf.}$$
$$= \frac{1}{2(1-\cos x)} = \frac{1}{4\sin\frac{1}{2}x^2}.$$

Propterea ſumto x pro ſemicircumferentia fiet $\cos x = -1$, $\cos 2x = 1$, $\cos 3x = -1$, $\cos 4x = 1$, &c., & conſequenter inventa ſeries tranſit in I.^{am}

$$1 - 2 + 3 - 4 + 5 - 6 + \&c. \ldots = \frac{1}{4}. \text{ Q. E. D.}$$

II. Lemmatis I. ſeries bis differentietur, dividaturque per dx^2, quo facto invenietur

(N) $-\cos x - 2^2\cos 2x - 3^2\cos 3x - 4^2\cos 4x - 5^2\cos 5x - \&c. = 0.$

Quocirca ſumto x pro ſemicircumferentia oritur ſeries II.^{a}

$1 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \&c. \ldots = 0.$

III. Capiatur ex aequatione (M) differentiale ſecundum, hocque per $-dx^2$ dividatur; quod dat aequationem

(O) $$-\cos x - 2^3\cos 2x - 3^3\cos 3x \ldots - \&c.$$
$$= \frac{-1 - 2\cos\frac{1}{2}x^2}{8\sin\frac{1}{2}x^4};$$

quae in praedicta ipſius x aſſumtione convertitur in ſeriem III.^{am}

$$1^3 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \&c. \ldots = -\frac{1}{8}.$$

IV. Eodem modo capta aequationis (N) differentia ſecunda, eaque diviſa per $-dx^2$ oritur aequatio

(P) $-\cos x - 2^4\cos 2x - 3^4\cos 3x - 4^4\cos 4x - 5^4\cos 5x \ldots \&c. = 0;$

haecque in hypotheſi $x = 180°$, abit in formulam IV.^{am}

$$1^4 - 2^4 + 3^4 - 4^4 + 5^4 - 6^4 + \&c. = 0$$

V. Aequatio (O) bis differentiata, ac per $-dx^2$ diviſa praebet

$$(Q) \quad -\text{coſ}\,x - 2^5\text{coſ}\,2x - 3^5\text{coſ}\,3x - 4^5\text{coſ}\,4x - \&c. = \frac{2\,\text{ſin}\,\frac{1}{2}x^2 + 13\,\text{coſ}\,\frac{1}{2}x^2 + 2\,\text{coſ}\,\frac{1}{2}x^4}{8\,\text{ſin}\,\frac{1}{2}x^5}$$

Hinc capto de more $x = 180°$, praedicta aequatio convertitur in formulam V.^{am}

$$1^5 - 2^5 + 3^5 - 4^5 + 5^5 - 6^5 \;\ldots\; + \&c. = -\frac{1}{4}$$

VI. Sumo aequationis (P) differentiam ſecundam, eamque divido per $-dx^2$, hincque aequationem nanciſcor

$$-\text{coſ}\,x - 2^6\text{coſ}\,2x - 3^6\text{coſ}\,3x - 4^6\text{coſ}\,4x \;\ldots\; - \&c. = 0.$$

Haec autem per ſubſtitutionem ſemicircumferentiae loco x degenerat in formulam VI.^{am}

$$1^6 - 2^6 + 3^6 - 4^6 + 5^6 - 6^6 \;\ldots\; + \&c. = 0. \quad \text{Q. E. D}$$

Hac ipſa methodo protinus demonſtrantur Theoremata omnia Euleriana circa ſeries poteſtatum poſitivarum integrarum ex numeris naturalibus cum ſignis alternantibus; & generatim conſtitui ratumque haberi poteſt, ſeries quantitatum parium praeditas eſſe ſumma improprie dicta, ſeu potius *quantitate genitrice*, ſemper nihilo aequali; ſeries vero poteſtatum imparium donari ſumma, vel *magnitudine genitrice*, ſemper a nihilo diverſa.

Theoremata alia Eulerianis analoga demonſtrari pariter poſſent circa ſeries poteſtatum numerorum imparium alternis ſignis affectarum

$$1^n - 3^n + 5^n - 7^n + 9^n - 11^n + \&c.$$

In hiſce numerorum imparium ſeriebus, ſecus atque in illis numerorum naturalium, illud generatim conſtitui poteſt *poteſtates impares orivi a magnitudine genitrice nihilo aequali; pares a magnitudine determinata*.

ADNO-

ADNOTATIO ad §. 208. *PART. II.*

PEr calculum differentialem inſigni compendio inveniri poſſunt ſequenti etiam modo coefficientes terminorum in formula, quae exprimit generalem terminum ſeriei cuiuſcunque recurrentis. Sit igitur ex ſerie recurrenti ordinis t

$$y_x,\ y_{x+1},\ y_{x+2}, \ldots y_{x+t-1},\ y_{x+t}$$

orta aequatio

$$A y_x + B y_{x+1} + C y_{x+2} \ldots + M y_{x+t-1} + y_{x+t} = 0,$$

ex qua facto $y_x = a z^x$, per ſubſtitutionem, elicitur (facta diviſione per $a z^x$) aequatio relationis

(A) $\qquad A + B z + C z^2 \ldots + M z^{t-1} + z^t = 0.$

Concipiamus nunc huiuſce aequationis radices eſſe $\alpha, \beta, \gamma, \delta$, &c., ſintque radices $= \alpha$ numero n: praedicta aequatio induet hanc formam

(B) $(z-\alpha)^n (z^m + p z^{m-1} + q z^{m-2} \ldots + f z^2 + g z + k) = 0 = P$

aſſumto $m + n = t$. Statuatur $z^m + p z^{m-1} + q z^{m-2} \ldots f z^2 + g z + k = Z$, eritque $(z-\alpha)^n Z = P$. Fiat porro $dZ = Z' dz$, $ddZ = Z'' dz$, $d^3 Z = Z''' dz$, &c. Hinc habebitur $\frac{dP}{dz} = n(z-\alpha)^{n-1} Z + (z-\alpha)^n Z'$ ubi facto $n = 1, z = \alpha$ prodit $\frac{dP}{d\alpha} = Z$. Differentiatio ſecunda praebet

$$\frac{ddP}{dz^2} = n(n-1)(z-\alpha)^{n-2} Z + 2n(z-\alpha)^{n-1} Z' + (z-\alpha)^n Z'';$$

ubi poſito $n = 2$, $z = \alpha$, evadit $\frac{ddP}{2 d\alpha^2} = Z$. Pari ratiocinatione invenietur $\frac{d^n P}{1.2.3 \ldots n d\alpha^n} = Z$, quando $z = \alpha$, & n numerus aliquis eſt ſeriei naturalis 1, 2, 3, &c. Iam vero eſt $Z = (z-\beta)(z-\gamma)(z-\delta)$ &c., & poſito α pro z, fit

$$Z = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) \text{ \&c.}$$

Ergo ſi in aequatione (B) unica eſt radix α, ſeu ſi $n = 1$, habemus $\frac{dP}{d\alpha} = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)$ &c.; ſi binae ſunt aequales radices α, ita ut ſit $\beta = \alpha$, habemus $\frac{d^2P}{2d\alpha^2} = (\alpha-\gamma)(\alpha-\delta)$ &c.;

Si tres ſunt radices aequales, hoc eſt $\alpha = \beta = \gamma$, nanciſcimur

$$\frac{d^3P}{2.3\,d\alpha^3} = (\alpha-\delta)(\alpha-\varepsilon) \text{ \&c. Sicque deinceps.}$$

2. Fiat modo $B + Cz + Dz^2 \ldots\ldots + z^{t-1} = Q$, & ob $A + Bz + Cz^2 \ldots + Mz^{t-1} + z^t = (z-\alpha)^n(z^m + pz^{m-1} + qz^{m-2} \ldots + fz^2 + gz + k)$, adeoque $A = k(-\alpha)^n$, ſublato A, factaque diviſione per z prodibit $Q = \frac{(z-\alpha)^n Z - k(-\alpha)^n}{z}$, ubi poſito $n = 1$, & $z = \alpha$ oritur $Q = k$. Ex differentiatione autem elicitur $\frac{dQ}{dz} = \frac{n(z-\alpha)^{n-1}Z + (z-\alpha)^n Z'}{z} - \frac{(z-\alpha)^n Z + k(-\alpha)^n}{z^2}$, ubi aſſumto $n = 2$, $z = \alpha$ prodit $\frac{dQ}{d\alpha} = k$. Si iterum differentietur; tum tertio; deinde quarto &c. inveniemus iugiter $\frac{ddQ}{2\,d\alpha^2} = k$, quando $n = 3$, $z = \alpha$; pariter $\frac{d^3Q}{2.3\,d\alpha^3} = k$, quando $n = 4$, $z = \alpha$, & in univerſum $\frac{d^{n-1}Q}{1.2.3\ldots(n-1)\,d\alpha^{n-1}} = k$ exiſtente n numero aliquo ſeriei naturalis 2, 3, 4, &c. a binario incipientis. Sed k aequatur producto ex $-\beta \times -\gamma \times -\delta$ &c. Igitur huic producto differentialia praedicta aequabuntur.

Statuatur $R = C + Dz + Ez^2 \ldots\ldots + z^{t-2}$; eritque ob $A = k(-\alpha)^n$, $B = nk(-\alpha)^{n-1} + g(-\alpha)^n$, demtis ex aequatione (A) terminis $A + Bz$, ac diviſione facta per z^2, $R = \frac{(z-\alpha)^n Z - z[nk(-\alpha)^{n-1} + g(-\alpha)^n]}{z^2}$; quae ſane

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expreſſio in hypotheſi $n=1$, $z=\alpha$, migrat in $R=g$. Capto differentiali, factoque $n=2$, $z=\alpha$, invenitur $\frac{dR}{d\alpha}=g$; ſicque rurſus ſumto $n=3$, $z=\alpha$ detegitur $\frac{d^2R}{2d\alpha^2}=g$, &c. eodem quo antea tenore atque ordine. Eſt autem g aggregatum productorum ex radicibus $-\beta, -\gamma, -\delta$, &c. quae producta denominationem mutuantur a numero $n-1$. Igitur huic productorum aggregato aequantur differentialia omnia praedicta.

Pari modo ſi ponatur $S=D+Ez\ldots+z^{r-1}$, invenietur $\frac{d^{n-1}S}{1.2.3\ldots(n-1)d\alpha^{n-1}}=f$, ideſt $=$ aggregato productorum ex radicibus $-\beta, -\gamma, -\delta$, &c., quae producta denominationem accipiunt ab exponente $n-2$. Ita ſemper de ceteris.

3. Sit iam exempli cauſſa ſeriei recurrentis terminus generalis $y_x=a\alpha^x+b\beta^x+c\gamma^x$, ubi (*Adnot. poſt Cap. II.*)

$$a=\frac{\beta\gamma y_0-(\beta+\gamma)y_1+y_2}{(\alpha-\beta)(\alpha-\gamma)};\ b=\frac{\alpha\gamma y_0-(\alpha+\gamma)y_1+y_2}{(\beta-\alpha)(\beta-\gamma)};$$

$$c=\frac{\alpha\beta y_0-(\alpha+\beta)y_1+y_2}{(\gamma-\alpha)(\gamma-\beta)}$$

Quoniam in hac hypotheſi tres habentur radices, eaeque inaequales α,β,γ, erit idcirco $\frac{dP}{d\alpha}=(\alpha-\beta)(\alpha-\gamma)$; $Q=\beta\gamma$; $R=-\beta-\gamma$.

Quapropter prodibit $a=\frac{Qy_0+Ry_1+y_2}{\frac{dP}{d\alpha}}$, qui valor migrat in b mutando α in β, & β in α; pariterque migrat in c permutando α cum γ, & γ cum α.

4. Concipiamus nunc in allato exemplo binas eſſe radices aequales $\alpha=\beta$, in qua hypotheſi terminus generalis hanc induit formam $a'\alpha^x+b'\alpha^{x-1}+c\gamma^x$, ubi (*loco cit.* §. 16.)

$a'=$

$$a' = \frac{-y_2 + 2\alpha y_1 + (\gamma^2 - 2\alpha\gamma) y_0}{(\alpha-\gamma)^2};\ b' = \frac{y_2 - (\alpha+\gamma) y_1 + \alpha\gamma y_0}{\alpha - \gamma}$$

$$c = \frac{y_2 - 2\alpha y_1 - \alpha\alpha y_0}{(\gamma-\alpha)^2}.$$

Valorem a' revoco ad hanc formam $\frac{-\gamma y_0 + y_1}{\alpha-\gamma} + \frac{-y_2 + (\alpha+\gamma) y_1 - \alpha\gamma y_0}{(\alpha-\gamma)^2}$ quae a praecedenti non differt, uti liquet. In hac itaque hypotheſi, in qua $n = 2$, invenitur $Q = \frac{-k\alpha^2}{\alpha} = -\alpha k = \alpha\gamma$, ob $k = -\gamma$. Pariter facto $n = 2$ detegitur $R = \frac{-\alpha^2 k - g\alpha^3 + 2\alpha^3 k}{\alpha^2} = k - \alpha g = -\gamma - \alpha$, ob $k = -\gamma, g = 1$. Quocirca deducimus $b' = \frac{Qy_0 + Ry_1 + y_2}{\frac{ddP}{2d\alpha^2}}$. Eſt autem manifeſto $\frac{-\gamma y_0 + y_1}{\alpha-\gamma} = \frac{\frac{dQ}{d\alpha} y_0 + \frac{dR}{d\alpha} y_1}{\frac{ddP}{2d\alpha^2}}$, & rurſus

$$\frac{-y_2 + (\alpha+\gamma) y_1 - \alpha\gamma y_0}{(\alpha-\gamma)^2} = \frac{-b'}{\alpha-\gamma} = -\frac{b'}{\frac{ddP}{2d\alpha^2}}.$$ Igitur

$$a' = \frac{-\gamma y_0 + y_1}{\alpha-\gamma} + \frac{-y_2 + (\alpha+\gamma) y_1 - \alpha\gamma y_0}{(\alpha-\gamma)^2} = \frac{\frac{dQ}{d\alpha} y_0 + \frac{dR}{d\alpha} y_1 - b'}{\frac{ddP}{2d\alpha^2}}.$$

Hinc coefficientium indeterminatorum valores ſequenti modo exprimentur

$$\text{I. } c = \frac{y_2 - 2\alpha y_1 + \alpha\alpha y_0}{(\gamma-\alpha)^2};\ \text{II. } b' = \frac{Qy_0 + Ry_1 + y_2}{\frac{ddP}{2d\alpha^2}};\ \text{III. } a' = \frac{\frac{dQ}{d\alpha} y_0 + \frac{dR}{d\alpha} y_1 - b'}{\frac{ddP}{2d\alpha^2}}$$

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5. Eodem omnino ratiocinandi modo ſi tres ſint radices aequales, & in termino ſeriei generali ſint c'', b'', a'' coefficientes indeterminati aequalibus radicibus praefixi (quaecunque demum fuerint radices reliquae inaequales) ſequentes formulas ſemper praeſto habebimus

$$\text{I. } c'' = \frac{Qy_0 + Ry_1 + Sy_2 + Ty_3 \text{ \&c.}}{\frac{d^3P}{2.3\,d\alpha^3}};$$

$$\text{II. } b'' = \frac{\frac{dQ}{d\alpha}y_0 + \frac{dR}{d\alpha}y_1 + \frac{dS}{d\alpha}y_2 \text{ \&c.} - c''}{\frac{d^3P}{2.3\,d\alpha^3}};$$

$$\text{III. } a'' = \frac{\frac{ddQ}{2d\alpha^2}y_0 + \frac{ddR}{2d\alpha^2}y_1 + \frac{ddS}{2d\alpha^2}y_2 \text{ \&c.} - b''}{\frac{d^3P}{2.3\,d\alpha^3}}$$

Atque ex his perſpicue patet progreſſus ad caſus alios radicum aequalium quatuor, quinque, ſex &c.

ADNOT. ad CAP. XI. PART. II.

1. Functio aliqua quantitatis cuiuſcunque variabilis tunc ſemper *Maxima* evadit vel *Minima*, cum eius functionis differentialia ordinum ſucceſſivorum ſimul evaneſcentia numero impari ſunt; & *Maxima* quidem, quotieſcunque eius differentiale ultimo evaneſcenti ſuperſtes negativum eſt; *Minima* ſi poſitivum. Demonſtratur hoc ab Auctore, aliiſque paſſim.

2. Hiſce conſtitutis, ac probe perpenſis ſit modo Z algebraica functio quantitatum variabilium t, u, x, y, &c., cu-

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ius *maximos*, *minimoſve* valores indagare oportet; erit itaque ex demonſtratis $dZ = pdt + qdu + rdx + sdy +$ &c. unde protinus haec manat aequatio

$$pdt + qdu + rdx + sdy + \&c. = 0 .$$

Quum porro relatio inter variabiles t, u, x &c., nec non inter ipſarum differentialia dt, du, dx &c. ſit adhuc indeterminata, & praedicta aequatio locum ſemper habere debeat, quaecunque ſit huiuſmodi relatio; hinc evidenter conſequitur, ſingula membra pdt, qdu, rdx &c. nihilo aequari oportere, proindeque tot oriri aequationes, quot exiſtunt variabiles, nimirum I. $p = 0$; II. $q = 0$; III. $r = 0$; &c. Harum aequationum ſubſidio inveniuntur quantitatum incognitarum t, u, x, &c. valores, qui in functione Z ſubrogati eam reddunt *Maximam*, vel *Minimam*.

3. Progrediamur nunc ad ſecundum differentiale. Aſſumtis, quod licet, conſtantibus primis differentialibus dt, du, dx &c. prodibit $d^2Z = dpdt + dqdu + drdx + dsdy +$ &c. Sit

$$dp = Adt + Bdu + Ddx + Gdy$$
$$dq = Bdt + Cdu + Edx + Hdy$$
$$dr = Ddt + Edu + Fdx + Idy$$
$$ds = Gdt + Hdu + Idx + Ldy$$

Ex quo eruitur

$$d^2Z = Adt^2 + 2\,Bdtdu + Cdu^2 + 2\,Ddtdx + 2\,Edudx + Fdx^2 + 2\,Gdtdy + 2\,Hdudy + 2\,Idxdy + Ldy^2 .$$

Ut a caſu omnium ſimpliciſſimo ordiamur, ponamus unam tantum eſſe variabilem t, ideoque $d^2Z = Adt^2$, ubi, ob valorem dt^2 ſemper affirmativum, differentiale d^2Z eodem afficietur ſigno, quo quantitas A: quocirca *minima* erit Z, ſi A fuerit affirmativa: *maxima* ſi A negativa; inventaque $A = 0$, a ſubſequentibus ipſius Z differentialibus petendum erit, ut liquet, *Maximi Minimive* criterium.

4. Ponamus nunc, in functione Z variabiles duas t & u contineri. In hac hypotheſi oritur

$$d^2Z = Adt^2 + 2Bdtdu + Cdu^2 = A\left(dt + \frac{Bdu}{A}\right)^2 + \left(C - \frac{B^2}{A}\right)du^2.$$

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In hac aequatione, quoniam quadrata $\left(dt + \frac{Bdu}{A}\right)^2$ & du^2 ſemper affirmativa ſunt, differentiale ſecundum d^2Z neceſſario affirmativum erit quotieſcunque coefficientes bini A, & $C - \frac{B^2}{A}$ fuerint affirmativi; contra vero negativum erit, ſi iidem fuerint ambo negativi, quaecunque ſit differentialium dt & du mutua relatio. Igitur pro *Minimo* functionis Z valore habebitur $A > o$, $C - \frac{B^2}{A} > o$, nimirum $C > \frac{B^2}{A}$, ſeu $CA > B^2$, unde infertur $C > o$. Quamobrem functio propoſita Z *Minima* eſſe nequit niſi tres ſequentes ſimul habeantur conditiones.

I. $A > o$; II. $C > o$; III. $AC > B^2$.

Pari ratiocinatione inveniemus pro *Maximo* ipſius Z valore fieri oportere $A < o$; $C - \frac{B^2}{A} < o$, ſeu $C < \frac{B^2}{A}$; ideoque $CA > B^2$, ob A ſcilicet negativum; ex quo etiam infertur $C < o$; quocirca functio Z valorem *Maximum* nanciſci non poteſt, niſi tres ſimul conditiones locum habeant:

I. $A < o$; II. $C < o$; III. $CA > B^2$.

Hinc porro patet, *Maximi* conditiones partim congruere, partim adverſari conditionibus *Minimi*.

5. Si vel A, vel C, vel utraque ſimul ſit $= o$, quin tamen ſit quoque $B = o$, conditio praecedens $AC > B^2$ ſtare haudquaquam poteſt; proindeque propoſita quantitas valorem *Maximum Minimumve* nunquam aſſequetur. Idem prorſus eveniet ſi A & C ſigna habuerint contraria; tunc enim ob valorem B^2 ſemper affirmativum conditio $AC > B^2$ impoſſibilis evadit. Si B evaneſcat una cum A, vel una cum C, ſecundi differentialis d^2Z valor ab unica variabili penderet, & conſequenter vel *Maximus* vel *Minimus*, vel neuter eſſet iuxta criteria ab Auctore tradita pro functionibus unius variabilis. Denique ſi tota quantitas d^2Z ſit $= o$, nimirum $A = o$, $B =$

$B = 0$, $C = 0$, ad tertium differentiale $d^3 Z$ conſugere oportet, quo non evaneſcente functio Z nec *Maxima* nec *Minima* eſſe poteſt, eo autem evaneſcente, differentiale quartum $d^4 Z$ quaerendum erit, cuius valor an affirmativus, an potius negativus fuerit, per methodum hic expoſitam facile dignoſcetur; ex quo rurſus *Maximus* vel *Minimus* functionis propoſitae valor colligetur.

6. Si Z functio ſit trium variabilium t, u, x, differentiale $d^2 Z$ induit hanc formam

$$d^2 Z = A dt^2 + 2 B dt du + C du^2 + 2 D dt dx + 2 E du dx + F dx^2$$

$$= A\left(dt + \frac{B du}{A} + \frac{D dx}{A} \right)^2 + \left(C - \frac{B^2}{A} \right) du^2$$

$$+ 2\left(E - \frac{BD}{A} \right) du dx + \left(F - \frac{D^2}{A} \right) dx^2$$

Fiat $C - \frac{B^2}{A} = a$; $E - \frac{BD}{A} = b$; $F - \frac{D^2}{A} = c$,

& aequatio tranſibit in hanc

$$d^2 Z = A\left(dt + \frac{B du}{A} + \frac{D dx}{A} \right)^2 + a du^2 + 2 b du dx + c dx^2$$

$$= A\left(dt + \frac{B du}{A} + \frac{D dx}{A} \right)^2 + a\left(du + \frac{b dx}{a} \right)^2 + \left(c - \frac{b^2}{a} \right) dx^2$$

Iam vero ob valorem quadratorum

$$\left(dt + \frac{B du}{A} + \frac{D dx}{A} \right)^2, \left(du + \frac{b dx}{a} \right)^2, \; \& \; dx^2$$

ſemper affirmativum, differentiale $d^2 Z$ erit pariter affirmativum ſi coëfficientes A, a, & $c - \frac{b^2}{a}$ ſigno + afficientur. Igitur pro *Minimo* functionis Z valore conditiones habentur ſequentes $A > 0$; $a > 0$; $c a > b^2$; ſeu ipſarum a, b, c valoribus ſubrogatis

$$A > 0; \; C - \frac{B^2}{A} > 0; \; \left(C - \frac{B^2}{A} \right)\left(F - \frac{D^2}{A} \right) > \left(E - \frac{BD}{A} \right)^2;$$

nimirum $A > 0$; $CA > B^2$; & (CA

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$$(CA-B^2)(FA-D^2)>(EA-BD)^2;$$

unde iterum oritur $C>o$, $F>o$, & $FA>D^2$. Quocirca functionis Z *Minimum* quinque hiſce conditionibus definietur I. $A>o$; II. $C>o$; III. $F>o$; IV. $CA>B^2$; V. $FA>D^2$.

Pari ratiocinatione inveniemus, valorem *Maximum* ipſius Z conditiones poſtulare

$A<o$, $CA>B^2$, & $(CA-B^2)(FA-D^2)>(EA-BD)^2$;

nam I. debet eſſe $a<o$, nimirum $C-\frac{B^2}{A}<o$, vel $C<\frac{B^2}{A}$;

eſt autem A negativum; igitur $CA>B^2$: II. eſt $c-\frac{b^2}{a}<o$;

hoc eſt $c<\frac{b^2}{a}$; proindeque ob a negativum $ca>b^2$, hoc eſt

$$\left(C-\frac{B^2}{A}\right)\left(F-\frac{D^2}{A}\right)>\left(E-\frac{BD}{A}\right)^2$$

vel $(CA-B^2)(FA-D^2)>(EA-BD)^2$;

unde oritur $FA>D^2$, & $F<o$. Quocirca *Maximi* conditiones ſunt quae ſequuntur

I. $A<o$; II. $C<o$; III. $F<o$; IV. $CA>B^2$; V. $FA>D^2$.

7. Si quantitates A & C vel ſingulae, vel ambae evaneſcunt IV. conditio ſit impoſſibilis. Si evaneſcat F corruit conditio V.: Ex hiſce conſequitur functionem Z nec *Maximam*, nec *Minimam* eſſe poſſe quotieſcunque quantitates A, C, F vel ſingulae vel omnes evaneſcant.

Ex dictis luculenter innoteſcit theoriam hanc ad functiones quatuor, vel etiam plurium variabilium aptari.

8. Quoniam haec nova theoria EULERO eſt prorſus intacta, non inutile erit ſequentia animadvertere.

Quicunque ſit numerus variabilium, quae functionem propoſitam Z ingrediuntur, ſi earum quaelibet ſeorſim ſpectetur, quaeraturque *Maximum* vel *Minimum*, quod eidem conveniat dum interea reliquae eaedem perſeverant, invenientur ſingillatim differentialia prima pdt, qdu, rdx, sdy &c., quorum

rum unumquodque nihilo aequatum aequationes §. 2. expoſitas praebebit, nimirum $p = 0$, $q = 0$, $r = 0$, $s = 0$ &c. Eodem modo ad ſecunda differentialia progredientes inveniemus ſeorſim quantitates Adt^2, Cdu^2, Fdx^2, Ldy^2 &c. ; ubi ſi A, C, F, L &c. ſint vel omnes affirmativae, vel omnes negativae facile inde quiſpiam colligeret valores ipſarum t, u, x &c. ex aequationibus $p = 0$, $q = 0$, &c. elicitos neceſſario inducere functioni propoſitae Z valorem *Maximum Minimumve*. Et ſane exploratum eſt quam quod maxime, functionem ipſam Z habita tantum ratione unius cuiuslibet ex praedictis variabilibus *Maximam* vel *Minimam* eſſe oportere: at quis tuto pronunciabit, id, quod pro variabili qualibet ſeorſim ſpectata obtinet, pro omnibus ſimul ſumtis obtinere? Rem penitius perſcrutemur.

9. Sit itaque Z functio duarum variabilium t & u, exhibeatque ordinatam ſuperficiei, cuius t & u ſunt ordinatae reliquae; ita ut ſuperficiei ipſius coordinatae ſint variabiles tres Z, t, & u. Iam quaeſtio eo redibit, ut inveniatur ordinata *Maxima* ſuperficiei, cuius datur aequatio, nimirum $dZ = pdt + qdu$. Facta u conſtanti aequatio abit in $dZ = pdt$, tuncque exprimit ſectiones omnes eiuſdem ſuperficiei parallelas axi quantitatum t, prouti quantitas u varios ſubinde recipit valores. Ponatur itaque $p = 0$; & ex hac aequatione is colligetur quantitatis t valor, qui in quablibet parallelarum ſectionum ordinatam Z *Maximam* redet aut *Minimam*. Et quoniam u conſtans aſſumitur, fiet differentiale ſecundum $d^2Z = Adt^2$, & conſequenter ex ſolo ipſius A valore licebit, de *Maximi* aut *Minimi* exiſtentia pronunciare, dummodo valor t ex aequatione $p = 0$ petitus ſubſtituatur. Nimirum ſi pro quolibet ipſius u valore quantitas A negativa inveniatur vel poſitiva, ſectiones praedictae omnes *Maximum* habebunt aut *Minimum*; & ſi pro vario ipſius u valore quantitas A ſignis diverſis afficiatur, ſectiones eaedem intra certos ſtatoſque limites *Maximo* donabuntur, intra alios *Minimo*. Si fuerit $A = 0$, quicunque

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ſit conſtantis u valor, tunc ſectionum illarum nulla *Maximum* habebit aut *Minimum*. At ſi A tunc ſolum fiat $= 0$, cum u certos definitoſque valores habet, in hac hypotheſi eae tantum ſectiones, quae praedictis ipſius u valoribus reſpondent, *Maximo Minimove* deſtituentur. Omnium harum ordinatarum locus geometricus continetur in aequatione $p = 0$, ſola variabilitate ipſius u ſpectata, ideoque ordinatae ipſae in eadem ſuperficie ſectionem conſtituent, quae ſimplicis erit, vel duplicis curvaturae, quaeque determinabitur per aequationes binas coniunctas $dZ = pdt + qdu$, & $p = 0$, ſeu $dZ = qdu$, & $p = 0$. Ex quo perſpicuum fit ad invenienda totius ſuperficiei *Maxima* vel *Minima*, quaeri oportere ordinatam *Maximam* vel *Minimam* huic ipſi ſectioni convenientem; unde iterum habebitur $q = 0$, quae porro aequatio valorem alterius variabilis u quaeſito ſatisfacientem ſtatim exhibebit.

10. Tranſeamus nunc ad differentiale ipſius q, nimirum ad aequationem prius inventam $dq = Bdt + Cdu$. Quum itaque ex aequatione $p = 0$ determinetur t per u, ſive in illius aequationis differentiali $Adt + Bdu = 0$ ſit $dt = -\frac{Bdu}{A}$, hoc ſubrogato valore nanciſcimur $dq = \left(-\frac{B^2}{A} + C\right) du$: ex quo conſequitur, ordinatam *Minimam* fore, ſi quantitas $-\frac{B^2}{A} + C$ poſitiva fuerit, ideſt $C > \frac{B^2}{A}$; *Maximam* vero ſi $C < \frac{B^2}{A}$; nec *Maximam* denique nec *Minimam* ſi $C = \frac{B^2}{A}$, dummodo reliqua altiora differentialia conditionibus ſupra indicatis teneantur. Iam vero *Maxima* huiuſmodi vel *Minima* ſerio perpendentes facili negotio comperimus, ordinatam Z ceterarum omnium *Maximam* eſſe non poſſe, niſi ſimul ſit illarum omnium *Maxima*, quae in ſectione per aequationem $dZ = qdu$ determinata continentur, & niſi praeterea ordinatae omnes

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hanc ſectionem componentes ſint totidem *Maxima* in ſectionibus parallelis homologis. Pari ratiocinatione quantitas Z nequit eſſe *Minima*, quin ſit itidem *Minima* in ſectione, quae *Minima* omnia complectitur. Ex hoc porro licet colligere, valores ipſarum t & u ex aequationibus $p = 0$, $q = 0$ deductos, & in quantitatibus A, & $C - \frac{B^2}{A}$ ſubſtitutos, ordinatae Z valorem *Maximum* inducere, ubi A fuerit negativa, & $C < \frac{B^2}{A}$, hoc eſt $CA > B^2$; & contra ordinatae Z per illam ſubſtitutionem valorem *Minimum* induci, ubi fuerit A affirmativa, & $C > \frac{B^2}{A}$, vel $CA > B^2$; quae omnia cum generali theoria ſuperius explicata uſquequaque conſentiunt.

11. Conditiones praedictae locum habent quando u ſpectatur primo conſtans, & t tanquam variabilis: at ſi contra variabilis accipiatur u & conſtans t, ad ſequentes conditiones pervenietur: $C < 0$, & $AC > B^2$ pro *Maximo*: $C > 0$, & $AC > B^2$ pro *Minimo*; quod ſane eodem redit. Ceterum methodus haec altera inveniendi *Maximorum Minimorumque* conditiones in functionibus duarum tantum variabilium, aeque accommodatur functionibus aliis magis complexis, eſtque praeterea magis analytica & directa quam methodus prior. Quamobrem non inutile erit eam hic generaliter evolvere.

12. Sint variabiles in functione Z contentae quot libuerit: ceteris habitis ut conſtantibus earum unam tantum ut variabilem reputo, & ex differentiatione aequationem pro *Maximo* aut *Minimo* deduco; tum ſumto in hac ipſa hypotheſi ſecundo differentiali, conditiones determino, quae functioni Z valorem *Maximum* vel *Minimum*, vel neutrum inducunt. Hac prima operatione abſoluta ſubſtituo in functione Z vel in eius differentialibus valorem primae variabilis inventum, eodemque modo in variabili altera calculum inſtituo, & va-

& valorem huius ſecundae variabilis ex differentiatione profectum in functione Z ſubſtituo: deinde ad examen tertiae variabilis progredior, ſicque porro eodem ſemper tenore. Sit t prima variabilis, quae ſpectari poteſt in Z, eritque $dZ = pdt$; $d^2Z = Adt^2$; ex quo $p = 0$, & $A > 0$ pro *Minimo*; $A < 0$ pro *Maximo*. Sint modo t & u ambae variabiles, orieturque $dZ = pdt + qdu$, quae aequatio ob $p = 0$ abit in $dZ = qdu$, unde profluit $d^2Z = (Bdt + Cdu)\,du$, ſed ob $p = 0$ eſt etiam $dp = 0$, & conſequenter $Adt + Bdu = 0$, ſeu $dt = -\frac{B\,du}{A}$, atque hic valor in aequatione praecedenti ſubrogatus eandem reddit $d^2Z\left(-\frac{B^2}{A} + C\right)du^2$. Erit igitur $q = 0$; & $-\frac{B^2}{A} + C > 0$ pro *Minimo*; $-\frac{B^2}{A} + C < 0$ pro *Maximo*; cumque aliàs ſit A poſitiva pro *Minimo*, negativa pro *Maximo*, eadem ſemper pro utroque exurget conditio $AC > B^2$.

Si praeter duas praecedentes tertia quoque variabilis x conſideranda veniat, tunc quaeritur ipſius dZ valor habita trium variabilium t, u, x ratione, oriturque $dZ = pdt + qdu + rdx$, quae aequatio ob $p = 0$, $q = 0$ tranſit in $dZ = rdx$. Quocirca habebitur ſecundum differentiale $d^2Z = (Ddt + Edu + Fdx)\,dx$; iam vero ex aequationibus $p = 0$, $q = 0$, vel $dp = 0$, $dq = 0$, hoc eſt $Adt + Bdu + Ddx = 0$, & $Bdt + Cdu + Edx = 0$ quaero valores ipſorum dt & du per dx expreſſos & invenio

$$dt = \frac{BE - CD}{AC - B^2}\,dx;\quad du = \frac{BD - AE}{AC - B^2}\,dx;$$

Hiſque ſubrogatis in expreſſione d^2Z pervenio ad aequationem $d^2Z = \left(\frac{BE - CD}{AC - B^2}\,D + \frac{BD - AE}{AC - B^2}\,E + F\right)dx^2$. Hinc igitur conſequitur pro *Maximo* vel *Minimo* haberi primo $r = 0$; deinde $\frac{BE - CD}{AC - B^2}\,D + \frac{BD - AE}{AC - B^2}\,E + F > 0$ pro *Minimo*,

& < 0

& < 0 pro *Maximo*: quocirca ſublato denominatore $AC - B^2$, qui ſemper poſitivus eſt, orietur

$2BDE - CD^2 - AE^2 - FB^2 + ACF > 0$ pro *Minimo*, & < 0 pro *Maximo*. Ducatur haec expreſſio in A, qui poſitivus eſt in primo caſu, negativus in altero, & prodibit

$$2ABDE - ACD^2 - A^2E^2 - AB^2F + A^2CF > 0$$

tam pro *Maximo* quam pro *Minimo*, ideſt

$$(CA - B^2)(FA - D^2) > (AE - BD)^2.$$

Quantumvis magnus fuerit variabilium numerus, eadem ſemper methodo res conficietur.

Novam hanc theoriam Clariſſimus De la Grange, a quo illam derivavimus, ſequentibus illuſtrat exemplis.

EXEMPLUM I.

Sint globi perfecte elaſtici quotcunque in recta linea diſpoſiti & a ſe mutuo ſeiuncti; incurrat primus velocitate data c *in ſecundum quieſcentem, ſecundus acquiſita velocitate incurrat in tertium, hic in quartum, atque ita deinceps ad poſtremum uſque; datis iam primi & ultimi maſſis quaeruntur maſſae omnium intermediorum ad hoc, ut poſtremus velocitatem omnium* maximam *ex impactu acquirat.*

Sit primi maſſa a, & poſtremi b, ſintque t, u, x, y &c. maſſae intermediae incognitae, ex notis impactus legibus velocitas, quam primus globus a alteri t communicat, invenitur $= \frac{2ac}{a+t}$; velocitas, quam hic ſecundus imprimit tertio x, detegitur $= \frac{2act}{(a+t)(t+u)}$; velocitas quarti x ex impactu tertii prodit $= \frac{2actu}{(a+t)(t+u)(u+x)}$; ſicque porro de ceteris. Quamobrem velocitas ultimi b exprimetur per

$$\frac{2\,tatuxy\;\ldots\;.\,b}{(a+t)(t+u)(u+x)(x+y)\;\ldots\;.},$$

quae quantitas *Maxima* ſit oportet. Eam pono $= Z$, & ſumtis

tis utrinque logarithmis reperio $l\,2\,ca + lt + lu + lx + ly +$ &c. $- l(a+t) - l(t+u) - l(u+x) - l(x+y) -$ &c. $= l\,Z$, unde oritur per differentiationem

$$\left.\begin{array}{l} \frac{dt}{t} + \frac{du}{u} + \frac{dx}{x} + \frac{dy}{y} + \&c. \\ -\frac{dt}{a+t} - \frac{dt+du}{t+u} - \frac{du+dx}{u+x} - \frac{dx+dy}{x+y} - \&c. \end{array}\right\} = \frac{dZ}{Z},$$

collectiſque terminis eodem differentiali affectis; iiſque ad eundem denominatorem redactis invenitur

$$dZ = \frac{Z(au - t^2)dt}{t(a+t)(t+u)} + \frac{Z(tx - u^2)du}{u(t+u)(u+x)} + \frac{Z(uy - x^2)dx}{x(u+x)(x+y)} + \&c.$$

Hiſce peractis habemus pro *Maximo Minimove* aequationes ſequentes $au = t^2$; $tx = u^2$; $uy = x^2$; &c. quae praebent analogias $a : t :: t : u :: u : x :: x : y$ &c., videlicet $\div\, a : t : u : x : y : \ldots . b$, ex quo pronum eſt colligere, maſſas omnes globorum progreſſionem geometricam conſtituere binis extremis a & b circumſcriptam. Ut nunc criterium ſupra expoſitum uſurpantes *Maximum* a *Minimo* internoſcamus, fiat compendii gratia

$$\frac{Z}{t(a+t)(t+u)} = \alpha;\ \frac{Z}{u(t+u)(u+x)} = \beta;\ \frac{Z}{x(u+x)(x+y)} = \gamma\ \&c.$$

Igitur comparatione inſtituta cum aequatione §. 2. invenietur $p = \alpha(au - t^2)$; $q = \beta(tx - u^2)$; $r = \gamma(uy - x^2)$; &c. : ac propterea $dp = (au - t^2)d\alpha + \alpha(adu - 2tdt)$; $dq = (tx - u^2)d\beta + \beta(xdt + tdx - 2udu)$; $dr = (uy - x^2)d\gamma + \gamma(ydu + udy - 2xdx)$; &c. Cum autem termini a, t, u, x, y &c. ſint continue proportionales vocata $1 : m$ ratione conſtanti antecedentis cuiuslibet ad conſequentem ſuum, inveniemus $t = ma$, $u = m^2 a$, $x = m^3 a$, $y = m^4 a$ &c.; praeterea $\beta = \frac{\alpha}{m^3}$, $\gamma = \frac{\alpha}{m^6}$ &c.; qui porro valores in praecedentibus expreſſionibus ſubrogati dant

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$$dp = \alpha a(du - 2m\,dt)$$
$$dq = \alpha a\left(dt - \frac{2\,du}{m} + \frac{dx}{m^2}\right)$$
$$dr = \alpha a\left(\frac{du}{m^2} - \frac{2\,dx}{m^3} + \frac{dy}{m^4}\right);$$

atque ita deinceps. Erit igitur per ſupra demonſtrata

$$A = -2m\alpha a;\ B = \alpha a;\ C = -\frac{2\alpha a}{m};\ D = 0;\ E = \frac{\alpha a}{m^2};$$

$$F = -\frac{2\alpha a}{m^3};\ G = 0;\ H = 0;\ I = \frac{\alpha a}{m^4};\ \&c.$$

Hinc autem ſtatim apparet A negativam eſſe, & conſequenter ſi ceterae conditiones impleantur quantitatem

$$\frac{2catuxy\ldots\ldots b}{(a+t)(t+u)(u+x)(x+y)\ldots},$$ ſeu celeritatem globo ultimo impreſſam *Maximam* eſſe. Iam vero $AC = 4\alpha^2 a^2$, & $B^2 = \alpha^2 a^2$ quocirca invenitur I. $AC > B^2$.

Tum $AC - B^2 = 3\alpha^2 a^2$; $FA - D^2 = \frac{4\alpha^2 a^2}{m^2}$; $EA - BD = -\frac{2\alpha^2 a^2}{m}$;

$$(AC - B^2)(FA - D^2) = \frac{12\alpha^4 a^4}{m^2};\ \&\ (EA - BD)^2 = \frac{4\alpha^4 a^4}{m^2};$$

atque inde oritur II. $(AC - B^2)(FA - D^2) > (EA - BD)^2$. Si bini tantum ſint interpoſiti globi, primam duntaxat harum conditionum attendere ſufficiet; ſi tres fuerint interiecti globi, altera quoque conditio erit attendenda; ſicque pro variabilium numero conditionum pariter augebitur multitudo. Ceterum ſi labore improbo calculus longius protrahatur invenientur in hoc problemate conditiones omnes adamuſſim impletae, ita ut tuto liceat pronunciare, in globorum continuo proportionalium ſerie quacunque, velocitatem ultimo impreſſam per interpoſitorum actionem eſſe omnium poſſibilium *Maximam*.

Problema hoc ab Ugenio primum, poſtea a Geometris aliis tractatum nullis certis determinationibus regebatur, quas tamen

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men hic neceſſarias invenimus ad *Maximi* aut *Minimi* exiſtentiam, & diſtinctionem evincendam.

EXEMPLUM II.

Sit generalis aequatio ſuperficierum ſecundi ordinis

$$z^2 = ax^2 + 2bxy + cy^2 - ex - fy;$$

& inveniendum proponatur ſuperficiei punctum, in quo ordinata z *evadit omnium* Maxima, *aut* Minima.

Sumto aequationis differentiali detegitur

$$2zdz = (2ax + 2by - e)\,dx + (2bx + 2cy - f)\,dy;$$

indeque habentur aequationes binae ſequentes:

$$2ax + 2by - e = 0;\ 2cy + 2bx - f = 0,$$

quae praebent $x = \frac{ec - bf}{2(ac - b^2)}$; $y = \frac{af - eb}{2(ac - b^2)}$. Sumo differentiale ſecundum, & ob $dz = 0$, invenio

$$2zd^2z = 2adx^2 + 4bdxdy + 2cdy^2.$$

Iam vero ut ordinata z omnium *Maxima* ſit, quantitates a & c ambae negativae ſint oportet; & contra affirmativae ut ſit *Minima*. At ſi exiſtente licet hac conditione deeſſet conditio altera nimirum $ca > bb$, valores inventi x & y in aequatione ſuperficiei ſubſtituti ordinatam z haudquaquam *Maximam* vel *Minimam* redderent: Et ſane Auctor noſter in *Appendice Introductionis ad Analyſim Infinitorum Tom.* 2. ex aliis principiis dilucide oſtendit, ubi ca non eſt $> b^2$, ſuperficiem propoſitam in infinitum extendi, & aſymptoto conica donari. Id quod omnem reſpuit *Maximorum Minimorumve* conceptum.

13. Hinc luculenter apparet, vulgarem *Maximorum*, ac *Minimorum* methodum, per quam in quaeſtione plures variabiles involvente ſingulae ſeorſim ſpectantur, & indiviſa omnium conſideratio negligitur, pluribus ſaepe incommodis erroribuſque laborare. Et re quidem vera in Exemplo praecedenti ſi reſpicimus ſolam x tanquam variabilem, invenimus differentiale primum $2\left(ax + by - \frac{e}{2}\right)dx$, & ſecundum

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$2adx^2$; pariter ſola y variante habetur differentiale primum $2\left(cy+bx-\frac{f}{2}\right)dy$, & ſecundum $2cdy^2$. Bina haec prima differentialia nihilo aequata eaſdem praebent aequationes, quas praec. §. invenimus; & bina differentialia ſecunda indicant ordinatae z valorem *Maximum* aut *Minimum*, quotieſcunque ambae a & c negativae, vel affirmativae fuerint; quod ſane criterium deficiente conditione altera $ca > bb$ mancum ac mendoſum eſſe iam demonſtravimus.

ADNOT. ad CAP. XV. PART. II.

IN hoc Capite, quod Bernoullianam regulam de inveniendo fractionis indeterminatae $\frac{0}{0}$ valore complectitur, nulla fit ab Auctore mentio caſus cuiuſdam memorabilis, in quo regula ipſa deficere videtur, quando ſcilicet repetitis in infinitum differentiationibus, in eandem ſemper expreſſionem $\frac{0}{0}$ indeterminatam incurritur. Exemplum habemus in formula $\frac{1+x}{1:l(1+x)}$, quae ubi $x=-1$ abit in $0.l0$, ſeu $l0=-\infty=-\frac{1}{0}$, tranſit in $-\frac{0}{0}$, vel $\frac{0}{0}$. Captis enim differentialibus numeratoris ac denominatoris praedictae formulae $\frac{1+x}{1:l(1+x)}$ prodit $\frac{d(1+x)}{d[1:l(1+x)]}=-(1+x)[l(1+x)]^2=-0.\infty^2$, rurſuſque $-(1+x)[l(1+x)]^2$ converſo in $\frac{-(1+x)}{1:[l(1+x)]^2}$ habebitur $d[-$

$\frac{d[-(1+x)]}{d[1:(l(1+x))^2]} = \frac{(1+x)[l(1+x)]^2}{2} = -\frac{0.\infty^2}{2}$; & ſic deinceps in infinitum, prodeunte nimirum ſemper valore indeterminato.

Hunc itaque ſcopulum declinabimus, expreſſionem $o\,l\,o$ ſpecie tenus indeterminatam, re tamen ipſa determinatam, hoc eſt nihilo aequalem eſſe oſtendendo; quod ita perficimus: pono $o\,l\,o = x = x\,l\,e = l\,e^x$ (ſumto nimirum e pro logarithmorum hyperbolicorum baſi). Igitur $l\,o^o = l\,e^x$; factoque regreſſu a logarithmis ad numeros erit $o^o = e^x$. Eſt autem ut conſtat $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} +$ &c. Ergo $o^o = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} +$ &c. Iam vero $o^o = 1$; nam $o^o = (a-a)^{n-n} = \frac{(a-a)^n}{(a-a)^n} = 1$.

Itaque $1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} +$ &c., cui aequationi ſatisfieri nullo modo poteſt, niſi fuerit $x = o$. Ex quo conſequitur, quantitatem $o\,l\,o$ nihilo aequalem eſſe. Sicque pariter licebit oſtendere productum ex infiniteſima qualibet quantitate ducta in logarithmum alterius cuiuslibet infiniteſimae non aliud eſſe niſi infiniteſimum (*a*).

Si cui haec demonſtratio de quantitate $o^o = 1$ non arridet hic ſequentem ſubrogare poterit ratiocinationem: Sit ω magnitudo infiniteſima, pariterque λ infiniteſima; aio fore $\omega^\lambda = 1 + z$, exiſtente z quantitate infinite parva ordinis cuiuſdam indeterminabilis. Nam nequit eſſe $\omega^\lambda = 1$, ſecus foret $\omega = 1$ quod eſt abſurdum: rurſus nequit eſſe $\omega^\lambda = 1 + a$ (ſumta a pro quantitate finita affirmativa); ſecus eſſet $\omega = (1+a)^{\frac{1}{\lambda}} =$ infinito altiſſimi ordinis, quod pariter abſurdum

(*a*) *De mirifica indole quantitatum* $1o, 1\infty$, *&c. V. Cl.* G. FONTANAE Diſquiſitiones Phiſico-Mathematicas, *Diſq. XIII.* de infinito logarithmico.

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dum immane eſt: demum nequit eſſe $\omega^{\lambda} = 1 - a$ (exiſtente $a < 1$); ſecus eſſet $\omega = (1-a)^{\frac{1}{\lambda}}$, nimirum aequalis legitimae fractioni, puta $\frac{f}{f+g}$ ad infinitam poteſtatem evectae, hoc eſt

$$\omega = \left(\frac{f}{f+g}\right)^n = \frac{f^n}{f^n + \frac{nf^n g}{f} + \frac{n^2 f^n g^2}{2f^2} + \frac{n^3 f^n g^3}{2.3f^3} + \&c.}$$

$$= \frac{1}{1 + \frac{ng}{f} + \frac{n^2 g^2}{2f^2} + \frac{n^3 g^3}{2.3f^3} + \frac{n^4 g^4}{2.3.4f^4} + \&c.}$$

quae evidenter quantitas eſt infinite parva ordinis altiſſimi; & dividendo per ω haberetur ûnitas aequalis magnitudini infinite parvae altiſſimi ordinis per infiniteſimam primi ordinis diviſae, hoc eſt aequalis quotienti ſemper infiniteſimo, quod hypotheſi repugnat. Igitur eſt $\omega^{\lambda} = 1 + z$, & z nihil aliud eſſe poteſt quam infinite parva magnitudo. Quocirca ω^{λ}, & conſequenter o^{o} non differt ab unitate niſi magnitudine infinite parva, ſeu evaneſcente, quae proinde tuto negligitur

Si ex aequationibus $o^{o} = 1^{o} = a^{o}$ quiſpiam colligeret facto regreſſu a numeris ad logarithmos fore $o\,l\,o = o\,l\,1 = o\,l\,a$, & diviſione per o facta eſſe etiam $l\,o = l\,1 = l\,a$, nimirum infinitum negativum aequari finito, atque adeo nihilo, is ſemetipſum captioſa ſubtilitate illaquearet.

At quis concederet quantitatem quamlibet datam dividi poſſe per abſolutum nihilum, tuncque praeſertim, cum dividendum ipſum nihilum eſt abſolutum, praebetque quantitatem $\frac{o}{o}$, magnitudinem ſcilicet ita indeterminatàm & vagam, ut nullum non referat valorem? Figmenta haec ſunt intellectus noſtri, quae niſi circumſpecto maturoque iudicio uſurpentur, atque intra legitimos veri rectique cancellos coerceantur, inextricabiles hallucinatiónes progignunt.

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Tom. II. 1751.

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Tom. III. 1753.

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Tom. VI. 1761.

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Tom. VIII. 1763.

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Tom. IX. 1764.

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$$\frac{dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}} = \frac{dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}}.$$

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Sectio ſecunda de principiis motus fluidorum.

Tom. XIV. P. II. 1770.

Methodus ex obſervato tranſitu Veneris per Solem inveniendi parallaxin Solis, ſive:
Expoſitio methodorum, cum pro determinanda parallaxi Solis ex obſervato tranſitu Veneris per Solem, tum pro inveniendis longitudinibus locorum ſuper terra ex obſervationibus eclipſium Solis, una cum calculis & concluſionibus inde deductis.

Tom. XV. 1771.

Solutio problematis, quo duo quaeruntur numeri, quorum productum tam ſumma, quam differentia eorum, ſive auctum ſive minutum fiat quadratum.
Obſervationes circa radices aequationum.
Problema algebraicum ob affectiones prorſus ſingulares memorabile.
De curva rectificabili in ſuperficie ſphaerica.
Sectio tertia de motu fluidorum, lineari potiſſimum aquae.
Genuina principia doctrinae de ſtatu aequilibrii & motu corporum tam perfecte flexibilium quam elaſticorum.
De ictu glandium contra tabulam exploſarum.

Tom. XVI. 1772.

De ſolidis, quorum ſuperficiem in planum explicare licet.
Methodus nova ac facilis calculum variationum tractandi.
Evolutio formulae integralis $\int x^{f-1} dx (lx)^{\frac{m}{n}}$ integratione a valore $x = 0$ ad $x = 1$ extenſa.
Problematis cuiuſdam geometrici prorſus ſingularis evolutio.
Conſiderationes cyclometricae.
Sectio quarta de motu aeris in tubis.
De perturbatione motus terrae ab actione Veneris oriunda.

Tom. XVII. 1773.

Problematis cuiuſdam Diophantaei evolutio.
Obſervationes circa bina biquadrata, quorum ſummam in duo alia biquadrata reſolvere liceat.
De variis integrabilitatis generibus.
Obſervationes circa aequationem differentialem: $y\,dy + My\,dx + N\,dx = 0$.

Conſideratio aequationis differentio-differentialis

$$(a+bx)\,ddx+(c+ex)\frac{x}{dx.dz}+(f+gx)\frac{zdx^2}{x^2}=0.$$

Exercitationes analyticae.
Digreſſio de traiectoriis tam orthogonalibus quam obliquangulis.
De colliſone corporum gyrantium.
De colliſione corporum pendulorum, tam obliqua, quam motu gyratorio perturbata.
De vera tautochrona in fluido.
De tautochrona in medio rariſſimo, quod reſiſtit in ratione multiplicata quacunque celeritatis.
Dilucidationes de tautochroniſmo.
De chordis vibrantibus diſquiſitio ulterior.
Animadverſiones in ſolutinnem Bernoullianam de motu cordarum ex duabus partibus diverſae craſſitiei compoſitarum. Tom. XVI. Nov. Com. p. 410.
De motu vibratorio chordarum ex partibus quotcunque diverſae craſſitiei compoſitarum.
De motu vibratorio chordarum craſſitie uteunque variabili praeditarum.
De motu vibratorio laminarum elaſticarum, ubi plures novae vibrationum ſpecies haſtenus non pertractatae evolvuntur.
De motu gravium citiſſimo ſuper curvis ſpecie datis.

Tom. XVIII. 1774.

Summatio progreſſionum.

$$\text{ſin}\,\phi^\lambda+\text{ſen}\,2\phi^\lambda+\text{ſen}\,3\phi^\lambda\ldots.+\text{ſin}.\,n\phi^\lambda,$$
$$\text{coſ}.\,\phi^\lambda+\text{coſ}.\,2\phi^\lambda+\text{coſ}.\,3\phi^\lambda\ldots.+\text{coſ}.\,n\phi^\lambda.$$

Nova ſeries infinita maxime convergens perimetrum elliptis exhibens.
Demonitrationes circa reſidua ex diviſione poteitatum per numeros primos reſultantia.
Nova ratio quantitates irrationales proxime exprimendi.
Solutio problematis de inveniendo triangulo, in quo rectae ex ſingulis angulis latera oppoſita biſecantes ſint rationales.
Reſolutio aequationis: $Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0$ per numeros tam rationales quam integros.
Inſignes proprietates ſerierum ſub hoc termino generali contentarum:

$$x=\tfrac{1}{2}\left(a+\frac{b}{\sqrt{k}}\right)(p+q\sqrt{k})^n+\tfrac{1}{2}\left(a-\frac{b}{\sqrt{k}}\right)(p-q\sqrt{k})^n.$$

De reſolutione irratinnalium per fractiones continuas, ubi ſimul nova quaedam ſingularis ſpecies minimi exponitur.
Determinatio motus oſcillatorii in Diſſertatione Cel. Dan. Bernoulli pertractati, ex primis Mechanicae principiis petita.
De preſſinne ponderis in planum cui incumbit.
De harmoniae veris principiis per ſpeculum muſicum repraeſentatis.

Nova

Nova methodus motus planetarum principalium ad tabulas aſtronomicas reducendi.

Diſquiſitio de lentibus obiectivis triplicatis, quae vel nullam confuſionem pariant, vel etiam datam confuſionem a reliquis lentibus ortam deſtruere valeant.

De applicatione lentium obiectivarum compoſitarum ad omnis generis teleſcopia; ubi agitur:

1. De perfectione teleſcopiorum primi generis, nullam imaginem realem continentium.
2. De perfectione teleſcopiorum ſecundi generis, ſeu aſtronomicorum, unicam imaginem realem continentium.
3. De perfectione teleſcopiorum tertii generis, duas imagines reales continentium.

Tom. XIX. 1775.

De valore formulae integralis: $\int\left(\frac{z^{m-1}\pm z^{n-m-1}}{1\pm z^{n}}\right)dz$ caſu quo poſt integrationem ponitur $z=1$.

De valore formulae integralis: $\int\left(\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1+z^{2\lambda}}\right)\frac{x}{dz}(\log z)^{\mu}$ caſu quo poſt integrationem ponitur $z=1$.

Nova methodus quantitates integrales determinandi.

Demonſtratio theorematis Newtoniani, de evolutione poteſtatum binomii, pro caſibus quibus exponentes non ſunt numeri integri.

Problema Diophantaeum ſingulare.

De tabula numerorum primorum uſque ad millionem & ultra continuanda, in qua ſimul omnium numerorum non primorum minimi diviſores exprimantur.

De oſcillationibus minimis penduli quotcunque ponduſculis onuſti.

De motu oſcillatorio binarum lancium ex libra ſuſpenſarum.

Explicatio motus oſcillatorii mirabilis in libra maiore obſervati.

De motu turbinatorio chordarum muſicarum, ubi ſimul univerſa theoria tam aequilibrii, quam motus corporum flexibilium ſimulque etiam elaſticorum breviter explicatur.

Commentatio hypothetica de periculo, a nimia cometae appropinquatione metuendo.

Tom. XX. 1776.

Solutio quorundam problematum Diophantaeorum.

Speculationes analyticae.

Obſervationes circa novum & ſingulare ſerierum genus.

Meditationes circa ſingulare ſerierum genus.

Formulae generales pro translatione quacunque corporum rigidorum.

Nova methodus motum corporum rigidorum determinandi.

Nnnnn Regu-

Regula facilis pro diiudicanda firmitate pontis aliuſve corporis ſimilis, ex cognita firmitate moduli.

De gemina methodo tam aequilibrium quam motum corporum flexibilium determinandi, & utriuſque egregio conſenſu.

De preſſione funium tenſorum in corpora ſubiecta eorumque motu a frictione impedito; ubi praeſertim methodus traditur, motum corporum tam perfecte flexibilium quam utcunque elaſticorum non in eodem plano ſitorum determinandi.

De preſſione funium tenſorum in corpora ſubiecta eorumque motu a frictione impedito. Diſſertatio altera.

De traiectu citiſſimo ſtellae per duos circulos almucantharat datos, pro qualibet elevatione poli.

De circulo maximo fixo in coelo conſtituendo, ad quem orbitae planetarum & cometarum referantur.

NOVA ACTA ACADEMIAE SCIENTIARUM IMPERIALIS PETROPOLITANAE.

Tom. I. 1777. P. I.

Réflexions ſur quelques nouvelles expériences optiques, communiquées à l'Académie par M. Wilſon.

Obſervationes in diſſertationem Ill. Dan. Bernoulli.

De formulis exponentialibus replicatis.

De repraeſentatione ſuperficiei ſphaericae ſuper plano.

De proiectione geographica Deliſliana in mappa generali Imperii Ruſſici uſitata.

De proiectione geographica ſuperficiei ſphaericae.

Vera theoria refractionis & diſperſionis radiorum lucis, rationibus & experimentis confirmata.

De figura quam ventus fluido ſtagnanti inducere valet.

Conſiderationes ſuper problemate aſtronomico in Comment. Vet. Tomo IV. pertractato.

De figura apparente annuli Saturni, pro eius loco quocunque reſpectu terrae.

De apparitione & diſparitione annuli Saturni.

Tom. I. 1777. P. II.

De integratione formulae: $\int \frac{dx\, lx}{\sqrt{(1-xx)}}$, ab $x=0$ ad $x=1$ extenſa.

De valore formulae integralis: $\int \frac{x^{a-1}dx(1-x)^b(1-x)^c}{lx \qquad 1-x^n}$ a termino $x=0$ uſque ad $x=1$ extenſa.

Novae demonſtrationes circa reſolutionem numerorum in quadrata.

Conſiderationes circa Brachyſtocronas.

Sur l'effet de la réfraction dans les obſervations terreſtres.
De motu oſcillatorio penduli cuiuſcunque, dum arcus datae amplitudinis abſolvit.
De theoria lunae ad maiorem perfectionis gradum evehenda.

Tom. II. 1778. P. I.

De corporibus regularibus per doctrinam ſphaericam determinatis; ubi ſimul nova methodus, globos ſive coeleſtes ſive terreſtres charta obducendi traditur.
Dilucidationes ſuper methodo elegantiſſima, qua Ill. de la Grange uſus eſt in integranda aequatione differentiali $\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$
De infinities infinitis gradibus tam infinite magnorum quam infinite parvorum.
Determinatio onerum, quae columnae geſtare valent.
Examen inſignis paradoxi in theoria columnarum occurrentis.
De altitudine columnarum ſub proprio pondere corruentium.
Réflexions ſur les inégalités dans le mouvement de la terre, cauſées par l'action de Venus; avec une table des corrections du lieu de la terre.
Inveſtigatio perturbationum, quae in motu terrae ab actione Veneris producuntur, cum tabula perturbationum illarum.

Tom. II. 1778. P. II.

De curvis triangularibus.
De menſura angulorum ſolidorum.
De caſibus quibuſdam maxime memorabilibus in analyſi indeterminata; ubi imprimis inſignis uſus calculi angulorum in analyſi Diophantaea oſtenditur.
De motu oſcillatorio duorum corporum ex filo ſuper trochleas traducto ſuſpenſorum.
De problemate quodam mechanico ſatis obvio, at ſolutu difficillimo.
Solutio gemina problematis, quo motus corporis, filo alicubi alligati, ſuper plano horizontali quaeritur.
Nova methodus motum planetarum determinandi.

Tom. III. 1779. P. I.

De formatione fractionum continuarum.
De tribus numeris quadratis, quorum tam ſumma quam ſumma productorum ex binis fit quadratum.
Trigonometria ſphaerica univerſa, ex primis principiis breviter & dilucide derivata.
De motu oſcillatorio mixto plurium pendulorum ex eodem corpore mobili ſuſpenſorum.
Inveſtigatio motuum, quibus laminae & virgae elaſticae contremiſcunt.
Coniectura circa naturam aeris, pro explicandis phaenomenis in atmoſphaera obſervatis.
Annotatio in Diſſertationem Cel. Kratzenſteinii, de Tubi iconantidiptici ſive

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duplicantis emendatione.
Theoria parallaxeos ad figuram terrae ſphaeroidicam accommodata.

Tom. III. 1779. P. II.

De ſerie Lambertina, plurimiſque eius inſignibus proprietatibus.
De motu oſcillatorio pendulorum ex filo tenſo dependentium.
Dilucidationes ſuper aliquot caſus aequilibrii difficiliores.
Determinatio omnium motuum, quos chorda tenſa & uniformiter craſſa recipere poteſt.
De proprietatibus triangulorum mechanicis.
De figura curvae elaſticae, contra obiectiones quaſdam Ill. d' Alembert.
Cautiones neceſſariae in determinatione motus planetarum obſervanda.

Tom. IV. 1780. P. I.

Supplementum calculi integralis. De integratione formularum irrationalium.
Nova methodus fractiones quaſcunque rationales in fractiones ſimplices reſolvendi.
Evolutio producti infiniti:

$$(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)$$

in ſeriem ſimplicem.
De mirabilibus proprietatibus numerorum pentagonalium.
Problematis cuiuſdam Pappi Alexandrini conſtructio.
De motu libero plurium corporum filis colligatorum ſuper plano horizontali.
De vi fluminis ad naves ſurſum trahendas applicanda.
De ſtatu aequilibrii maris a viribus ſolis & lunae ſollicitati.
Determinatio facilis orbitae cometae, cuius tranſitum per eclipticam bis obſervare licuit.
De variis motuum generibus, qui in ſatellitibus planetarum locum habere poſſunt.
De motibus maxime irregularibus, qui in ſyſtemate mundano locum habere poſſunt, una cum methodo huiuſmodi motus per temporis ſpatium quantumvis magnum proſequendi.

Tom. IV. 1780. P. II.

De ellipſi minima dato parallelogrammo rectangulo circumſcribenda.
Speculationes circa quaſdam inſignes proprietates numerorum.
De plurimis quantitatibus tranſcendentibus, quas nullo modo per formulas integrales exprimere licet.
De inductione ad plenam certitudinem evehenda.
Dilucidationes de motu chordarum inaequaliter craſſarum.
De motu penduli circa axem cylindricum fulcro datae figurae incumbentem mobilis, remota frictione. Diſſertatio prior.
De motu penduli circa axem cylindricum fulcro datae figurae incumbentem mobilis, habita frictionis ratione. Diſſertatio altera.

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Tom. V. 1781. P. I.

Nova methodus integrandi formulas differentiales rationales, ſine ſubſidio quantitatum imaginariarum.
De duplici geneſi tam epicycloidum quam hypocycloidum.
De curvis rectificabilibus in ſuperficie coni recti ducendis.
De mirabilibus proprietatibus unciarum quae in evolutione binomii ad poteſtatem quamcunque evecti occurrunt.
De oſcillationibus minimis funis libere ſuſpenſi.
De perturbatione motus chordarum ab earum pondere oriunda.
De perturbatione motus planetarum & cometarum.

Tom. V. 1781. P. II.

Plenior explicatio circa comparationem quantitatum in formula integrali $\int \frac{Zdz}{\sqrt{(1+mz^2+nz^4)}}$ contentarum, denotante Z functionem quamcunque rationalem ipſius z.
Uberior evolutio comparationis, quam inter arcus ſectionum conicarum inſtituere licet.
De numero memorabili in ſummatione progreſſ. harmonice naturalis occurrente.
De inſignibus proprietatibus unciarum binomii ad uncias quorumvis polynomiorum extenſis.
De effectu frictionis in motu volutorio.

MEMOIRES DE L'ACADEMIE ROYALE DES SCIENCES DE PARIS.

Année 1765.

Précis d'une Théorie générale de Dioptrique.

Année 1778.

Eſſai d'une Théorie de la réſiſtance qu'éprouve la proue d'un vaiſſeau dans ſon mouvement.

RECUEIL DES PIECES QUI ONT REMPORTE LES PRIX DE L'ACADEMIE ROYALE DES SCIENCES DE PARIS.

Tome II.

Meditationes ſuper Problemate nautico de implantatione malorum. 1727.

Tome IV.

Diſſertatio de igne, in qua eius natura & proprietates explicantur. 1738.

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Inquiſitio phyſica in cauſam fluxus & refluxus maris. 1740.

Tome V.

De obſervatione inclinationis magneticae diſſertatio. 1743.
Diſſertatio de magnete. 1744.

Tome VII.

Recherches ſur la queſtion des inégalités du mouvement de ſaturne & de Jupiter. 1748.
Recherches ſur les irrégularités de Jupiter. 1752.

Tome VIII.

De promotione navium ſine vi venti. 1753.
Inveſtigatio perturbationum, quibus planetarum motus ob actionem eorum mutuam afficiuntur 1756.
Examen des efforts qu' ont à ſoutenir toutes les parties d' un vaiſſeau dans le roulis & dans le tangage. 1759.

Tome IX.

Théorie de la Lune & ſpéciellement ſur l'équation ſéculaire. 1770.
Nouvelles recherches ſur le vrai mouvement de la Lune. 1772.

MISCELLANEA BEROLINENSIA.

Tom. VII. 1743.

Determinatio orbitae cometae Anno 1742. obſervati.
Theoremata circa reductionem formularum integralium ad circuli quadraturam.
De inventione integralium, ſi poſt integrationem variabili quantitati determinatus valor tribuatur.
Diſſertatio altera de ſummis ſerierum reciprocarum ex poteſtatibus numerorum naturalium ortarum.
De integratione aequationum differentialium altiorum graduum.

MEMOIRES DE L'ACADEMIE R. DE SCIENCES DE BERLIN.

Tome I. Année 1745.

Sur la lumière & les couleurs.
Sur le choc & la preſſion.

Sur la nature des moindres particules de la matière.
Sur de nouvelles tables aſtronomiques pour calculer la place du Soleil.
Sur le mouvement des noeuds de la Lune & ſur la variation de ſon inclinaiſon a l'écliptique.
Sur quelques propriétés des ſections coniques.
Sur le mouvement des corps flexibles.
Sur la force de percuſſion & de ſa véritable meſure.
Sur quelques propriétés des ſections coniques, qui conviennent à une infinité d'autres lignes courbes.

Tom. II. Année 1746.

Recherches phyſiques ſur la cauſe de la queve des Comètes, de la lumière boréale & de la lumière zodiacale.
Mémoire ſur l'effet de la propagation ſucceſſive de la lumière dans l'apparition tant des Planètes que des Comètes.
Mémoire ſur la plus grande équation des Planètes.

Tom. III. Année 1747.

Recherches ſur le mouvement des corps céleſtes en général.
Méthode pour trouver les vrais momens tant des nouvelles que des pleines lunes.
Méthode de trouver le vrai lieu géocentrique de la Lune par l'obſervation de l'occultation d'une étoile fixe.
Méthode de déterminer la longitude des lieux par l'obſervation d'occultations des étoiles fixes par la Lune.
Mémoire ſur la force des rames.
Réflexions ſur la dernière éclipſe du Soleil du 25 Juillet, de l'année 1748.
Sur la perfection des verres objectifs des lunettes.

Tom. IV. Année 1748.

Sur les vibrations des cordes.
Sur l'accord des deux dernières éclipſes du Soleil & de la Lune avec mes tables, pour trouver les vrais momens des pleni-lunes & novi-lunes.
Sur l'Atmoſphère de la Lune, prouvée par la dernière éclipſe annulaire du Soleil.
Sur le frottement des corps ſolides.
Sur la diminution de la réſiſtance du frottement.
Recherches ſur les plus-grands & les plus-petits qui ſe trouvent dans les actions des forces.
Réflexions ſur quelques loix générales de la nature, qui s'obſervent dans les effets des forces quelconques.
Sur une contradiction apparente dans la doctrine des lignes courbes.
Démonſtration ſur le nombre des points, où deux lignes d'un ordre quelconque peuvent ſe couper.
Réflexions ſur l'eſpace & le tems.

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Tom. V. Année 1749.

De la controverſe entre Mrs. Leibnitz & Bernoulli ſur les logarithmes des nombres négatifs & imaginaires .
Sur le point de rebrouſſement de la ſeconde eſpèce de Mr. le Marquis de l' Hôpital .
Recherches ſur les racines imaginaires des équations .
Recherches ſur la préceſſion des équinoxes, & ſur la nutation de l' axe de la terre .
De la parallaxe de la Lune , tant par rapport à ſa hauteur qu' à ſon azimuth , dans l' hypothèſe de la terre ſphéroïdique .

Tom. VI. Année 1750.

Découverte d' un nouveau principe de Mécanique .
Réflexions ſur les divers degrés de lumière du Soleil & des autres corps céleſtes .
Avertiſſement au ſujet des recherches ſur la préceſſion des équinoxes .
Recherches ſur l' origine des forces .
Lettre à Mr. Merian .

Tom. VII. Année 1751.

Harmonie entre les principes généraux de repos & de mouvement de Mr. de Maupertuis .
Sur le principe de la moindre action .
Examen de la diſſertation de Mr. le Prof. Koenig, inſérée dans les Actes de Leipzig , pour le mois de Mars , 1751.
Addition à cet examen .
Eſſai d' une démonſtration métaphyſique du principe général de l' équilibre .
Calcul de probabilité dans le jeu de rencontre .
Application de la machine hydraulique de Mr. Segner à toutes ſortes d' ouvrages , & de ſes avantages ſur les autres machines hydrauliques dont on ſe ſert ordinairement .
Recherches ſur une nouvelle manière d' élever de l' eau , propoſée par Mr. Demour .

Tom. VIII. Année 1752.

Sur le mouvement de l' eau par des tuyaux de conduite .
Diſcuſſion plus particulière de diverſes manières d' élever de l' eau par le moyen des pompes avec le plus grand avantage .
Maximes pour arranger le plus avantageuſement les machines deſtinées à élever de l' eau par le moyen des pompes .
Eſſai d' une explication phyſique des couleurs engendrées ſur des ſurfaces extrêmement minces .

Tom. IX. Année 1753.

Remarques ſur les mémoires de Mr. Dan. Bernoulli, qui roulent ſur la courbe

que

que fait une corde tendue miſe en vibration, & ſur les différentes eſpèces de vibration des cordes.
Principes de la Trigonométrie ſphérique tirés de la méthode des plus-grands & des plus-petits.
Elémens de Trigonométrie ſphéroïdique, tirés de la méthode des plus-grands & des plus-petits.
Examen d'une controverſe ſur la loi de réfraction des rayons de différentes couleurs, par rapport à la diverſité des milieux tranſparens, par leſquels ils ſont tranſmis.
Recherches ſur la véritable courbe que décrivent les corps jettés dans l'air ou dans un autre fluide quelconque.

Tom. X. Année 1754.

De la réfraction de la lumière en paſſant par l'atmoſphère, ſelon les divers degrés tant de la chaleur que de l'élaſticité de l'air.
Réflexions ſur un problème de Géométrie traité par quelques Géomètres, & qui eſt néanmoins impoſſible.
Recherches phyſiques ſur la diverſe refrangibilité des rayons de lumière.
Théorie plus complette des machines qui ſont miſes en mouvement par la réaction de l'eau.
De la variation de la latitude des étoiles fixes & de l'obliquité de l'écliptique.

Tom. XI. Année 1755.

Principes généraux de l'état d'équilibre des fluides.
Principes généraux du mouvement des fluides.
Continuation des recherches ſur la Théorie du mouvement des fluides.

Tom. XII. Année 1756.

Recherches plus exactes ſur l'effet des moulins à vent.
Expériences pour déterminer la réfraction de toutes ſortes de liqueurs tranſparentes.
Sur l'action des ſcies.
Expoſition de quelques paradoxes dans le calcul intégral.

Tom. XIII. Année 1757.

Recherches ſur la déclinaiſon de l'aiguille aimantée.
Sur la force des colonnes.
Regles générales pour la conſtruction des téleſcopes & des microſcopes, de quelque nombre de verres qu'ils ſoient compoſés.
Recherches ſur les lunettes à trois verres qui repréſentent les obiets renverſés.

Tom. XIV. Année 1758.

Recherches ſur la connoiſſance mécanique des corps.

Du mouvement de rotation des corps ſolides autour d'un axe variable.
Remarques générales ſur le mouvement diurne des Planetes.

Tom. XV. Année 1759.

De la propagation du ſon.
Supplément aux recherches ſur la propagation du ſon.
Continuation des recherches ſur la propagation du ſon.
Recherches ſur le mouvement de rotation des corps céleſtes.
Solution d'une queſtion curieuſe qui ne paroit ſoumiſe à aucune analyſe, ſur la marche du cavalier ſur l'échiquier.

Tom. XVI. Année 1760.

Recherches ſur le mouvement des rivières.
Recherches ſur la courbure des ſurfaces.
Recherches générales ſur la mortalité & la multiplication du genre humain.
Sur les rentes viagères.
Du mouvement d'un corps ſolide quelconque, lorſqu'il tourne autour d'un axe mobile.
Problême: un corps étant attiré en raiſon réciproque quarrée des diſtances vers deux points fixes donnés, trouver le cas où la courbe décrite par ce corps ſera algébrique.

Tom. XVII. Année 1761.

Remarques ſur un beau rapport entre les ſéries des puiſſances tant directes que réciproques.
Recherches ſur la confuſion des verres dioptriques cauſée par leur ouverture.
Recherches ſur les moyens de diminuer ou de réduire même à rien la confuſion cauſée par l'ouverture des verres.
Nouvelle manière de perfectionner les verres objectifs des lunettes.
Détermination du champ apparent, que découvrent tant les téloſcopes que les microſcopes.
Règles générales pour la conſtruction des téleſcopes & des microſcopes.
Sur la perfection des lunettes aſtronomiques qui repréſentent les objets renverſés.

Tom. XVIII. Année 1762.

Conſidérations ſur les difficultés qu'on rencontre dans l'exécution des verres objectifs délivrés de toute confuſion.
Recherches ſur les téleſcopes à réflexion & les moyens de les perfectionner.
Recherches ſur une autre conſtruction des téleſcopes à réflexion.
Sur la confuſion que cauſe dans les inſtrumens dioptriques la diverſe réfrangibilité des rayons.
Conſidérations ſur les nouvelles lunettes d'Angleterre de Mr. Dollond, & ſur le principe qui en eſt le fondement.
Sur les avantages des verres objectifs compoſés de deux verres ſimples.

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Remarques ſur l' effet du frottement dans l' équilibre .

Tom. XIX. Année 1763.

Nouvelle méthode de déterminer les dérangemens dans le mouvement des corps céleſtes, cauſés par leur action mutuelle .
Réflexions ſur les diverſes manières , dont on peut repréſenter le mouvement de la Lune .
Conſidérations ſur le Problème des trois corps .
Nouvelle manière de comparer les obſervations de la Lune avec la Théorie .
Du mouvement des abſides des Satellites de Jupiter .

Tom. XX. Année 1764.

Nouvelle méthode d' éliminer les quantités inconnues des équations .
Recherches ſur les microſcopes ſimples & ſur les moyens de les perfectionner .
Recherches ſur les microſcopes à trois verres , & les moyens de les perfectionner .
Sur l' avantage du banquier au jeu de Pharaon .
Conjecture ſur la raiſon de quelques diſſonances généralement reçues dans la muſique moderne .
Des lunettes à trois verres , qui repréſentent les objets debout .

Tom. XXI. Année 1765.

Sur la probabilité des ſequences dans la lotterie Génoiſe .
Sur le mouvement d' une corde qui au commencement n' a été ébranlée que dans une partie .
Eclairciſſemens plus détaillés ſur la génération & la propagation du ſon , & ſur la formation de l' écho .

Tom. XXII. Année 1766.

Conſtruction des objectifs compoſés de deux différentes ſortes de verre , qui ne produiſent aucune confuſion , ni par leur ouverture , ni par la différente réfrangibilité des rayons , avec la manière la plus avantageuſe d' en faire des lunettes .
Conſtruction des objectifs compoſés , propres à détruire toute la confuſion dans les lunettes .
Réflexions ſur la manière d' examiner la réfraction du verre par le moyen des priſmes .
Corrections néceſſaires pour la Théorie de la déclinaiſon magnétique , propoſée dans le XIII.e Volume des Mémoires de l' Académie .

Tom. XXIII. Année 1767.

Méthode pour porter les verres objectifs des lunettes à un plus haut degré de perfection .

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Tom. XXV. Année 1769.

Solution d' une queſtion trés-difficile dans le calcul des probabilités.

NOUVEAUX MÉMOIRES DE L' ACADÉMIE DES SCIENCES ET BELLES LETTRES DE BERLIN.

Année 1772.

Extrait d' une lettre de Mr. Euler à Mr. Bernoulli, concernant le mémoire imprimé parmi ceux de 1771. p. 318. p. 35.

Année 1776.

Extrait d' une lettre de Mr. Euler à Mr. Beguelin, concernant les nombres premiers.

ACTA ERUDITORUM LIPSIENSIA.

Anno 1733.

Conſtructio aequationum quarundam differentialium, quae indeterminatarum ſeparationem non admittunt.

Anno 1744.

Solutio problematis, menſ. Nov. 1743. propoſiti.

Anno 1773.

Novae demonſtrationes circa reſolutionem numerorum in quadrata.

JOURNAL LITTÉRAIRE DE L' ALLEMAGNE.

Mois de Janv. & Fevr. 1751.

Découverte d' une loi extraordinaire des nombres.

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MISCELLANEA TAURINENSIA.

Tom. II. 1760-1761.

Lettre à Mr. de la Grange, contenant des recherches ſur la propagation des ébranlemens dans un milieu élaſtique.

Tom. III. 1762-1765.

Eclairciſſemens ſur le mouvement des cordes vibrantes.
Recherches ſur le mouvement des cordes inégalement épaiſſes.
Recherches ſur l'intégration de l'équation différentio-différentielle.

$$\left(\frac{ddz}{dx^2}\right)=aa\left(\frac{ddz}{dx^2}\right)+\frac{b}{x}\left(\frac{dz}{dx}\right)+\frac{c}{x^2}z.$$

Recherches ſur les nouvelles lunettes de cinq & ſix verres, & ſur leur perfection ultérieure.

Obſervationes circa integralia talium formularum: $\int x^{p-1}dx(1-x)^{\frac{q-1}{n}}$, poſito poſt integrationem $x=1$.

DISSERTATIONES IN SOCIETATE SCIENTIARUM FLISSINGENSI. PARS IX.

Recherches ſur une nouvelle eſpéce de quarrés magiques.

IN CALENDARIO ASTRONOMICO BEROLINENSI An. 1783.

Theorie der Parallaxe für die ſphäroidiſche Geſtalt der Erde.

DISSERTATIONES IN SOCIETATE OECONOMICA PETROPOLITANA PARS VI.

Nachricht von einem neuem Mittel zur Bermehrung des Getreides

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DISSERTATIONES INEDITAE.

Solutio problematis difficillimi ex methodo tangentium inverſa.
De ſymptomatibus quatuor punctorum in eodem plano ſitorum.
De deſcenſu baculi ſuper cylindro.

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De motu globi heterogenei ſuper plano horizontali, eiuſque motu a frictione impedito.
De motu globi circa axem obliquum quemcunque gyrantis & ſuper plano horizontali incedentis.
Methodus motum corporum tam perfecte flexibilium quam utcunque elaſticorum non in eodem plano ſitorum determinandi. Diſſertatio prior.
Methodus motum corporum tam &c. Diſſertatio altera.
Conſideratio motus plane ſingularis, qui in filo perfecte flexili locum habere poteſt.
De traiectoriis reciprocis tam rectangulis quam obliquangulis.
Commentatio de curvis tractoriis.
Conſiderationes ſuper traiectoriis tam rectangulis quam obliquangulis.
De tractoriis compoſitis.

De miris proprietatibus curvae elaſticae, ſub aequatione: $y = \int \frac{xx\,dx}{\sqrt{(1-x^4)}}$ contentae.
Enodatio difficultatis ſuper figura terrae a vi centrifuga oriunda.

Speculationes ſuper formula integrali: $\int \frac{x^n\,dx}{\sqrt{(aa+2bx+cxx)}}$, ubi ſimul egregiae obſervationes circa fractiones continuas occurrunt.
Inveſtigatio curvarum quae ſimiles ſint ſuis evolutis primis, ſecundis, tertiis, vel ordinis cuiuſcunque.
Novae demonſtrationes circa diviſores numerorum formae $xx + nyy$.
De motu ſanguinis per arterias.
De motu oſcillatorio tabulae ſuſpenſae & a vento agitatae.
De transformatione ſeriei divergentis:

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + \&c.$$

in fractionem continuam.
De viribus centripetis ad curvas non in eodem plano ſitas deſcribendas requiſitis.

Evolutio formulae integralis: $\int dx\left(\frac{1}{1-x}+\frac{1}{lx}\right)$, a termino $x=0$ uſque ad $x=1$ extenſae.
De ſummatione ſerierum, in quibus terminorum ſigna alternantur.
Uberior explicatio methodi ſingularis integralia alias maxime abſcondita inveſtigandi.
Methodus facilis inveniendi integrale huius formulae:

$$\int \frac{dx}{x}\cdot\frac{x^{n+p} - 2x^n \operatorname{cof.}\zeta + x^{n-p}}{x^{2n} - 2x^n \operatorname{cof.}\theta + 1},$$

caſu quo poſt integrationem ponitur $x=1$ & $x=\infty$.
Innumera theoremata circa formulas integrales, quorum demonſtratio vires Analyſeos ſuperare videatur.
De ſummo uſu calculi imaginariorum in Analyſi.
De ſingulari ratione differentiandi & integrandi, quae in ſummis ſerierum occurrit.

Specimen ſingulare Analyſeos infinitorum indeterminato.

De innumeris generibus ſerierum maxime memorabilium, quibus omnium aequationum algebraicarum non ſolum radices ipſae, ſed etiam quaecunque earum poteſtates exprimi poſſunt.

Analyſis facilis & plana ad eas ſeries maxime abſtruſas perducens, quibus omnium aequationum algebraicarum non ſolum radices ipſae, ſed etiam quaevis eorum poteſtates exprimi poſſunt.

De multiplicatione angulorum per factores expedienda.

Methodus generalis inveſtigandi radices omnium aequationum per approximationem.

Innumerae aequationum formae ex omnibus ordinibus, quarum reſolutio exhiberi poteſt.

Nova demonſtratio, quod evolutio poteſtatum binomii Newtoniana etiam pro exponentibus fractis valeat.

De innumeris curvis algebraicis, quarum longitudo per arcus parabolicos metiri poteſt.

De innumeris curvis algebraicis, quarum longitudo per arcus ellipticos metiri poteſt.

De curvis algebraicis, quarum longitudo exprimitur hac formula integrali:

$$\int \frac{v^{m-1}\,dv}{\sqrt{(1-v^{2n})}}$$

De binis curvis algebraicis inveniendis, quarum arcus indefinite inter ſe ſint aequales.

Quatuor theoremata maxime notatu digna in calculo integrali.

De lineis rectificabilibus in ſuperficie ſphaeroidica quacunque geometrice ducendis.

De formulis differentialibus, quae per tres pluresve quantitates datas multiplicatae fiant integrabiles.

De termino generali ſerierum hypergeometricarum.

De iterata integratione, dum aliquis exponens pro variabili aſſumitur.

Variae conſiderationes circa ſeries hypergeometricas.

De duabus pluribusve curvis algebraicis, in quibus ſi a terminis fixis arcus aequales abſcindantur, eorum amplitudines datam inter ſe teneant rationem.

De methodo tangentium inverſa ad theoriam ſolidorum translata.

Methodus facilis ducendi radios oſculi curvarum, ex methodo maximorum ac minimorum petita.

De ſuperficie coni ſcaleni.

Plenior expoſitio ſerierum illarum memorabilium, quae ex unciis poteſtatum binomii formantur.

De integratione formulae $\int \frac{dx\sqrt{(1+x^4)}}{1-x^4}$, aliarumque ſimilium.

De vero valore huius formulae integralis $\int dx\left(l\frac{1}{x}\right)^n$ ab $x=0$ ad $x=1$.

Exercitatio analytica.

Comparatio formulae integralis: $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ ab $x=0$ ad $x=1$.

De quadratis magicis.
De motu trium corporum ſe mutuo attrahentium ſuper eadem linea recta.
De radicibus aequationis infinitae:

$$0 = 1 - \frac{xx}{n(n+1)} + \frac{x^4}{n(n+1)(n+2)(n+3)} - \frac{x^6}{n\ldots.(n+5)} + \&c.$$

Determinatio onerum, quae columnae geſtare valent.
Evolutio problematis, cuius Solutio analytica eſt difficillima, dum ſynthetica per ſe eſt obvia.
De ſingulari genere quaeſtionum Diophantearum, & methodo maxime recondita eas reſolvendi.
Exercitatio analytica, ubi imprimis ſeriei maxime generalis ſummatio traditur.
De curvis hyperbolicis, quae intra ſuas aſymtotas ſpatium finitum includunt.
Problema geometricum ob ſingularia ſymptomata imprimis memorabile.
Dilucidationes ſuper formulis, quibus ſinus & coſinus angulorum multiplorum exprimi ſolent, ubi ingentes difficultates diluuntur.
Obſervationes generales circa ſeries, quarum termini ſecundum ſinus vel coſinus angulorum multiplorum progrediuntur.
De integrationibus maxime memorabilibus ex calculo imaginariorum oriundis; cum ſupplemento.
De integrationibus difficillimis, quarum integralia tamen aliunde exhiberi poſſunt.
Evolutio formulae integralis: $\int \frac{dz(3+zz)}{(1+zz)\sqrt[4]{(1+zz+z^4)}}$, per logarithmos & arcus circulares.
De formulis differentialibus ſecundi gradus, quae integrationem admittunt.
De inſignibus proprietatibus formularum integralium praeter binas variabiles etiam earum differentialia cuiuſcunque ordinis involventium.
Ulterior diſquiſitio de formulis integralibus imaginariis.
Specimen integrationis abſtruſiſſimae, hac formula: $\int \frac{dx}{(1+x)\sqrt[4]{(2xx-1)}}$ contentae.
Integratio formulae cuiuſdam differentialis maxime irrationalis, quam tamen per logarithmos & arcus circulares expedire licet.
Formae generales differentialium, quae etſi nulla ſubſtitutione rationales reddi poſſunt, tamen integrationem per logarithmos & arcus circulares admittunt.
De inſigni uſu calculi imaginariorum in calculo integrali.
De caſibus quibus formulam $x^4 + kxxyy + y^4$ ad quadratum reducere licet.
Integratio ſuccincta formulae maxime memorabilis: $\int \frac{dz}{(3 \pm zz)\sqrt[3]{(1 \pm 3zz)}}$.
De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos & arcus circulares integrare licet.
Methodus facilis inveniendi ſeries per ſinus coſinusve angulorum multiplorum procedentes, quarum uſus in univerſa theoria Aſtronomiae eſt ampliſſimus.

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Memorabile genus formularum differentialium maxime irrationalium , quas tamen ad rationalitatem perducere licet .
Diſquiſitio ulterior ſuper ſeriebus ſecundum multipla cuiuſdam anguli progredientibus .
Evolutio formulae analyticae , cui theoria Aſtronomiae potiſſimum innititur .
De centro ſimilitudinis .
Solutio problematis maxime curioſi , quo inter omnes ellipſes , quae circa datum triangulum circumſcribi poſſunt , ea quaeritur , cuius area ſit omnium minima .
Problema geometricum : inter omnes ellipſes , quae per data quatuor puncta traduci poſſunt , eam definire , quae habet aream minimam .
Supplementum in Calculi Integralis Tom. I. Cap. VI.
De inventione longitudinis locorum ex obſervata altitudine fixarum .
Eclairciſſemens ſur le mémoire de Mr. de la Grange , (Miſc. Taur.) concernant la méthode de prendre le milieu entre les réſultats de pluſieurs obſervations .
Recherches ſur quelques intégrations rémarquables dans l' Analyſe des fonctions à deux variables , connues ſous le nom de différences partielles .
Intégration d'une eſpèce remarquable d' équation différentielle , dans l' Analyſe des fonctions à deux variables .
Inveſtigatio ſuperficierum , quarum normales , ad datum planum productae , ſint omnes inter ſe aequales .
Exempla quarundam memorabilium aequationum differentialium , quas adeo algebraice integrare licet , etiamſi nulla via pateat variabiles a ſe invicem ſeparandi .
Reſolutio formulae Diophanteae : $ab(maa+nbb)=cd(mcc+ndd)$ per numeros rationales .
Inveſtigatio trianguli , in quo diſtantiae angulorum ab eius centro gravitatis rationaliter exprimantur .
De corporibus cylindricis incurvatis .
Enodatio inſignis paradoxi in determinatione perturbationum motus planetarum occurrentis .
Speculationes ſuper area triangulorum ſphaericorum .
Utrum hic numerus 1000009 ſit primus , nec ne ? inquiritur .
Methodus generalior numeros quoſvis ſatis grandes perſcrutandi , utrum ſint primi nec ne .
Facillima methodus plurimos numeros primos praemagnos inveniendi .
De formulis ſpeciei $mxx+nyy$, ad numeros primos explorandos idoneis , earumque mirabilibus proprietatibus .
De variis modis numeros praegrandes examinandi utrum ſint primi , nec ne .
Obſervationes ſingulares circa aequationes differentiales lineares .
De formulis integralibus implicatis earumque reſolutione & transformatione .
Regula facilis problemata Diophantea per numeros integros expedite reſolvendi .
Additamentum ad diſſertationem Tom. XI. Nov. Comment. inſertam.
De evolutione poteſtatis polynomialis cuiuſcunq. : $(1+x+xx+x^3+\&c.)^n$.
Diſquiſitiones analyticae ſuper evolutione poteſtatis trinomialis $(1+xx+x^3)^n$,
Specimen transformationis ſingularis ſerierum .

Diſquiſitio coniecturalis ſuper formula integrali : $\int\frac{d\phi \cos. i\phi}{(a+b\cos.\phi)^n}$.

Demonſtratio inſignis theorematis numerici circa uncias poteſtatum binomalium.

Demonſtratio theorematis inſignis, per coniecturam eruti, circa integrationem formulae : $\int\frac{d\phi \cos. i\phi}{(1+aa-2a\cos.\phi)^{n+1}}$.

Methodus nova & facilis omnium aequationum algebraicarum radices non ſolum ipſas, ſed etiam quaſcunque earum poteſtates per ſeries concinnas exprimendi.

De reſolutione fractionum compoſitarum in ſimpliciores.

Methodus facilis inveniendi radium oſculi pro curvis non in eodem plano ſitis.

Integratio aequationis differentialis : $dy+yydx=\frac{A\,dx}{(a+2bx+cxx)^2}$

Accuratior evolutio problematis de linea breviſſima in ſuperficie quacunque ducenda.

De lineis curvis non in eodem plano ſitis, quae maximi vel minimi proprietate ſunt praeditae.

Methodus ſingularis reſolvendi aequationes differentiales ſecundi gradus.

De inſigni paradoxo, quod in analyſi maximorum & minimorum occurrit.

Solutio problematis ob ſingularia calculi artificia memorabilis.

De motu quodam maxime memorabili, ſatis quidem ſimplici, at ſolutu difficillimo.

Dilucidationes ſuper problemate quodam geometrico, olim a Iacobo Bernoulli tractato.

Solutio completa problematis de quadriſectione trianguli per duas rectas inter ſe normales.

De ſummatione ſerierum in hac forma contentarum :

$$\frac{a}{1}+\frac{a^2}{4}+\frac{a^3}{9}+\frac{a^4}{16}+ \&c.$$

Inveſtigatio quarundam ſerierum, quae ad rationem peripheriae circuli ad diametrum definiendam maxime ſunt accommodatae, cum additamento.

De novo genere talium ſerierum.

Solutio facilior problematis in Tomo XVIII. Comment. Nov. pertractati, circa triangulum, in quo rectae latera oppoſita biſecantes rationaliter exprimantur.

De reſolutione formulae integralis : $\int x^{m-1}dx(\Delta+x^n)^\lambda$ in ſeriem ſemper convergentem, ubi ſimul inſignia artificia circa ſerierum ſummationem explicantur.

Solutio quaeſtionis curioſae ex doctrina combinationum.

Solutio facilis problematis geometrici, quo quaeritur circulus, qui tres alios circulos datos tangat.

Solutio facilis problematis geometrici, quo quaeritur ſphaera, quae quatuor alias datas tangat.

De ſerie maxime memorabili, qua poteſtas binomialis quaecunque exprimi poteſt.

Methodus nova inveſtigandi omnes caſus, quibus hanc aequationem differentio-differentialem: $ddy(1-axx)-bxdxdy-cydx^2=0$. reſolvere licet.

De fractionibus continuis Walliſii.

De ſeriebus memorabilibus, quibus ſinus & coſinus angulorum multiplorum exprimere licet.

Commentatio in fractionem continuam, qua Illuſtris la Grange poteſtates binomiales expreſſit.

Analyſis facilis aequationem Riccatianam per fractionem continuam reſolvendi.

Dilucidationes in Capita poſtrema Calculi mei differentialis, de functionibus inexplicabilibus.

Nova Methodus ſummas ſerierum infinitarum per formulas differentiales inveſtigandi.

De eclipſibus ſolaribus in ſuperficie ſphaerae per proiectionem repraeſentandis.

De integralibus quibuſdam inventu difficillimis.

Geometrica & Sphaerica quaedam.

Inveſtigatio quadrilateri, in quo ſingulorum angulorum ſinus datam inter ſe teneant rationem; ubi artificia prorſus ſingularia in analyſi Diophantea occurrunt.

Solutio ſuccinta & elegans problematis, quo quaeruntur tres numeri tales, ut tam ſummae quam differentiae binorum fiant quadrata.

De tribus pluribuſve numeris inveniendis, quorum ſumma ſit quadratum, quadratorum vero ſumma biquadratum.

Solutio problematis Fermatiani, de duobus numeris quorum ſumma ſit quadratum, quadratorum vero ſumma biquadratum.

De binis formulis $xx+myy$, & $xx+nyy$ inter ſe concordibus & diſcordibus.

De inſigni promotione analyſis Diophanteae.

Reſolutio facilis quaeſtionis difficillimae, qua haec formula generalis:

$$vvzz(axx+byy)^2+\Delta xxyy(avv+bzz)^2$$

ad quadratum reduci poſtulatur.

Solutio problematis difficillimi, quo hae duae formulae:

$aaxx+bbyy$, & $aayy+bbxx$ quadrata reddi debent.

Inveſtigatio accuratior circa brachyſtochronas.

Inveſtigatio binorum numerorum formae: $xy(x^4-y^4)$, quorum productum ſive quotus ſit quadratum.

De motu oſcillatorio penduli circa axem cylindricum plano horizontali incumbentem.

De binis numeris, quorum ſumma, ſive aucta ſive minuta tam unius quam alterius quadrato, producat quadrata.

De momentis virium reſpectu axis cuiuſcunque inveniendi; ubi plura inſignia ſymptomata, circa binas rectas non in eodem plano ſitas, explicantur.

Methodus facilis omnium virium momenta reſpectu axis cuiuſcunque determinandi.

Dilucidationes circa binas ſummas duorum biquadratorum inter ſe aequales.

De reſolutione huius aequationis:

$$0=a+bx+cy+dxx+exy+fyy+gxxy+hxyy+ixxyy.$$

per numeros rationales.

Methodus nova & facilis formulas cubicas & biquadraticas ad quadratum reducendi.
Conſtructio manometri, denſitatem aeris quovis tempore accurate monſtrantis.
De vera brachyſtochrona, ſeu linea celerrimi deſcenſus, in medio reſiſtente.
De valoribus integralium, a termino variabilis $x = 0$ uſque ad $x = \infty$ extenſorum.
De problemate curvarum ſynchronarum, eiuſque imprimis *inverſo*.
Enodatio maximi paradoxi in problemate quodam mechanico occurrentis.
Methodus nova & generalis problema ſynchronarum inverſum aliaque eiuſdem generis reſolvendi.
De curvis, quarum radii oſculi tenent rationem duplicatam diſtantiae a puncto fixo, earumque mirabilibus proprietatibus.
De infinitis curvis algebraicis, quarum longitudo indefinita arcui elliptico aequatur.
De binis curvis algebraicis eadem rectificatione gaudentibus.
De infinitis curvis algebraicis, quarum longitudo arcui parabolico aequatur.
De curvis algebraicis, quarum omnes arcus per arcus circulares metiri licet.
Recherches ſur deux problèmes de l' Analyſe de Diophante.
Supplément au Problème de quatre nombres, dont la ſomme de deux faſſe toujours un nombre quarré.
Problematis ad analyſin infinitorum indeterminatam referendi ſolutio.
De aequationibus differentialibus cuiuſcunque gradus, quae denuo differentiatae integrari poſſunt.
Solutio problematum difficiliorum ad methodum tangentium inverſam pertinentium.
De unciis poteſtatum binomii earumque interpolatione.
Specimen aequationum differentialium indefiniti gradus, earumque integrationis.
Solutio problematis mechanici.
Solutio alius problematis mechanici non parum curioſi.
De caſibus, quibus formulam $x^4 \pm mxxyy + y^4$ ad quadratum reducere licet.
De problemate traiectoriarum orthogonalium ad ſuperficies tranſlato.
Solutio problematis analytici difficillimi.

FINIS.

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